outline

disorder vs. topology in band insulators (qualitative)

field theory perspective (a little less qualitative)

quantum criticality of the disordered TI

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physics at the edge (of topological quantum wires)

quantum critical transport

Weyl semimetal—disordered gapless topological matter

Topological insulators

Anderson insulators

Topological insulators

Anderson insulators

olyacetylene
1977

Polypyrrole
1979

Polyphenylene
1979

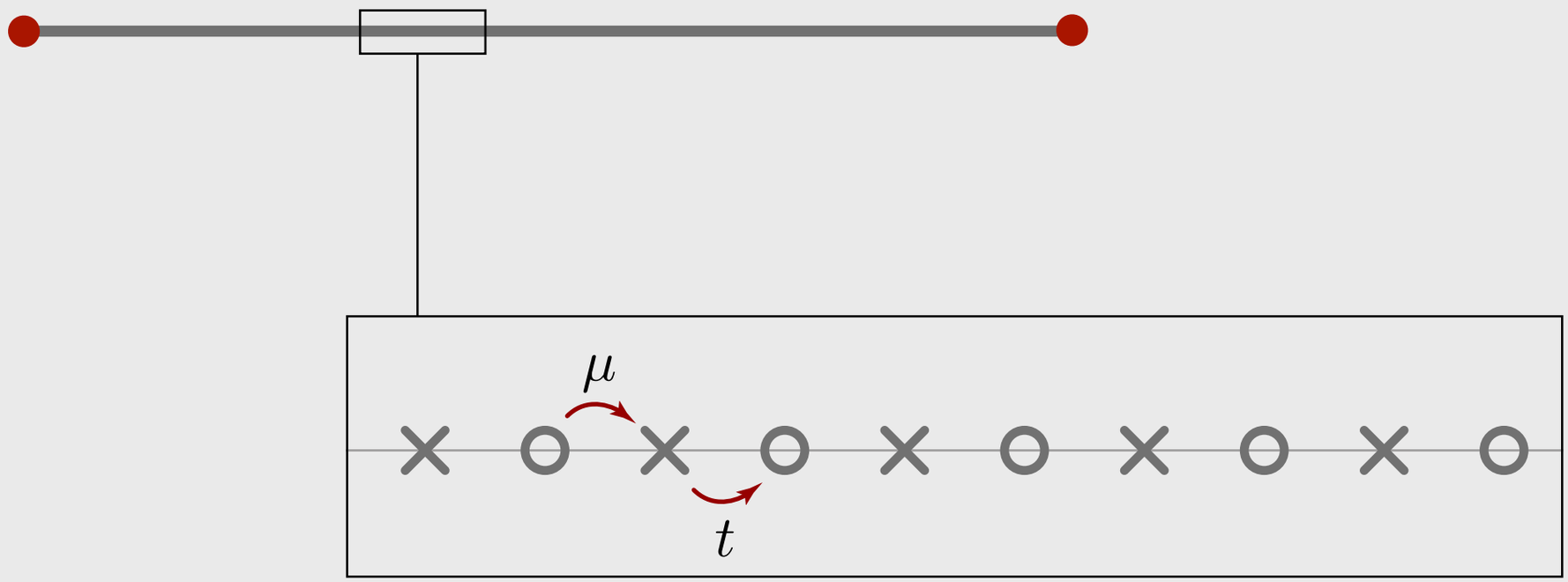
Poly(phenylene vinylene)
PPV 1979

Su-Schrieffer-Heeger chain

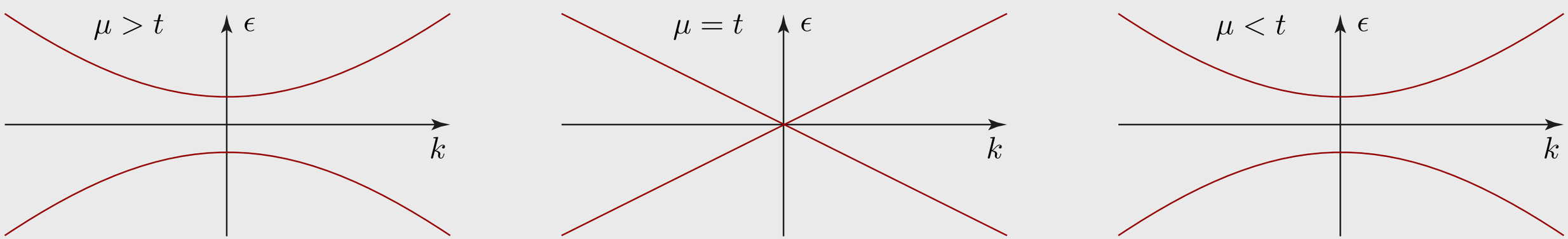
Polythiophene
1982

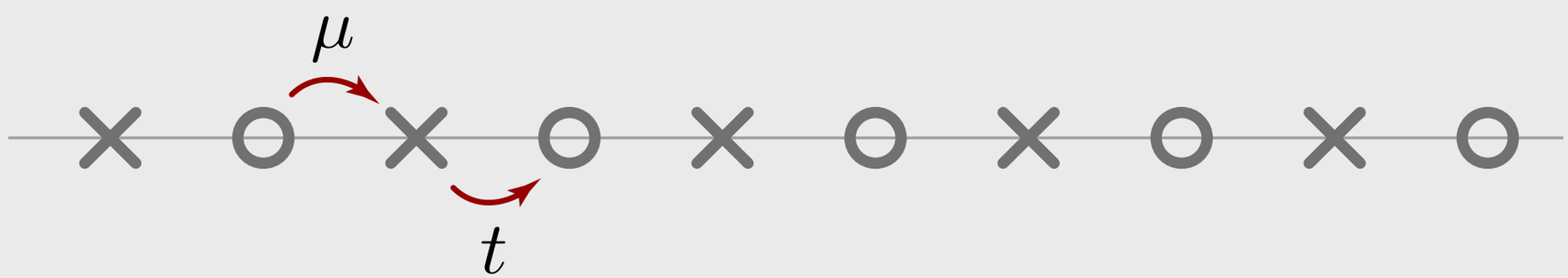
Polyaniline
1985

SSH chain

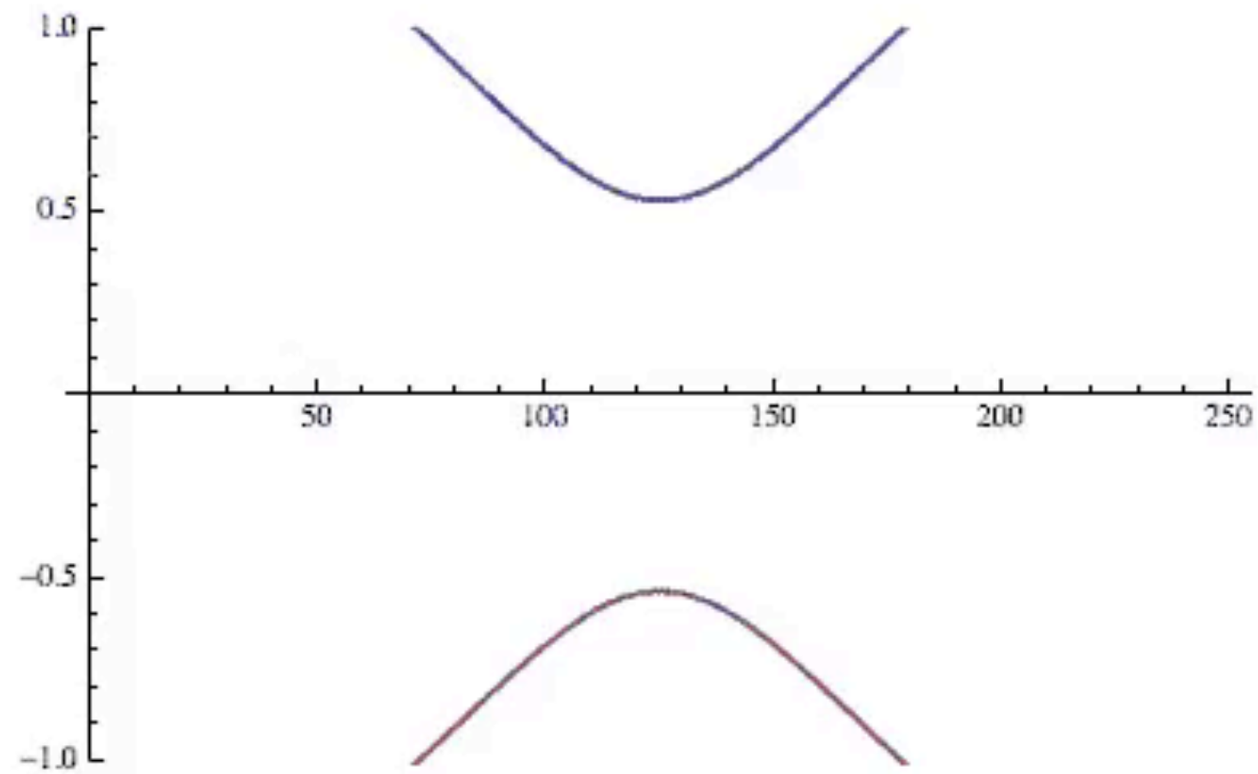


- $\mu > t$ system gapped
- $\mu < t$ system gapped & topologically charged

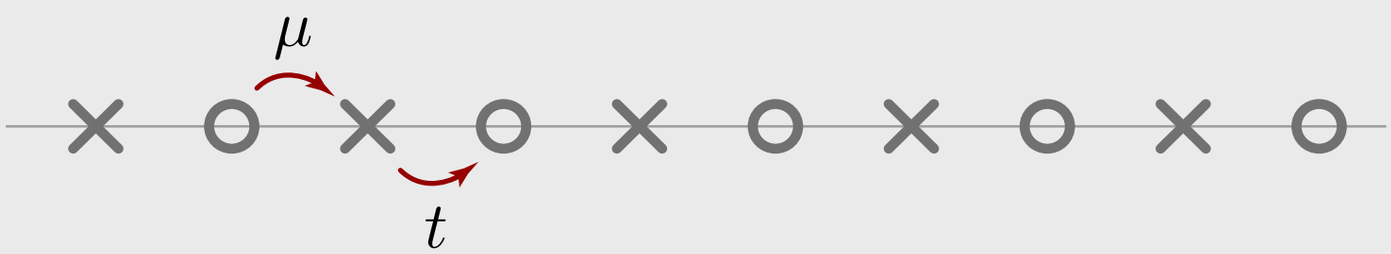



μ
t


0.20
0.15
0.10
0.05
0
50
100
150
200
250

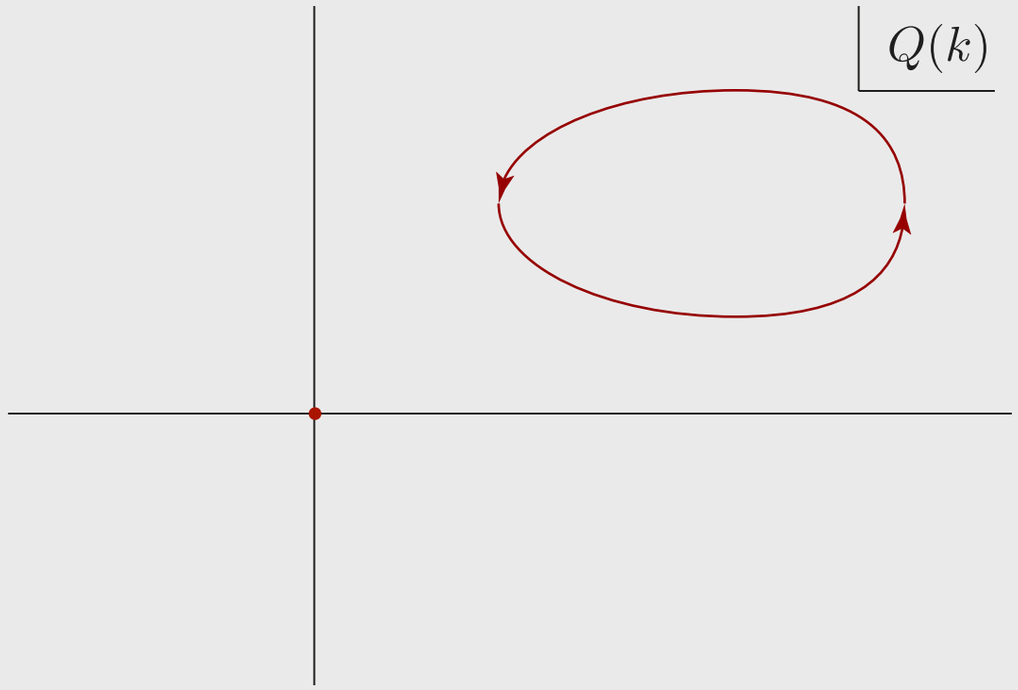

1.0
0.5
−0.5
−1.0
50
100
150
200
250

why topological?



$$\hat{H} = \begin{pmatrix} & \hat{Q} \\ \hat{Q}^\dagger & \end{pmatrix} \begin{matrix} \times \\ \circ \end{matrix}$$

- ▷ introduce Bloch crystal momentum $k \in [0, 2\pi]$
- ▷ $\hat{Q} \to Q(k) \in \mathbb{C}$

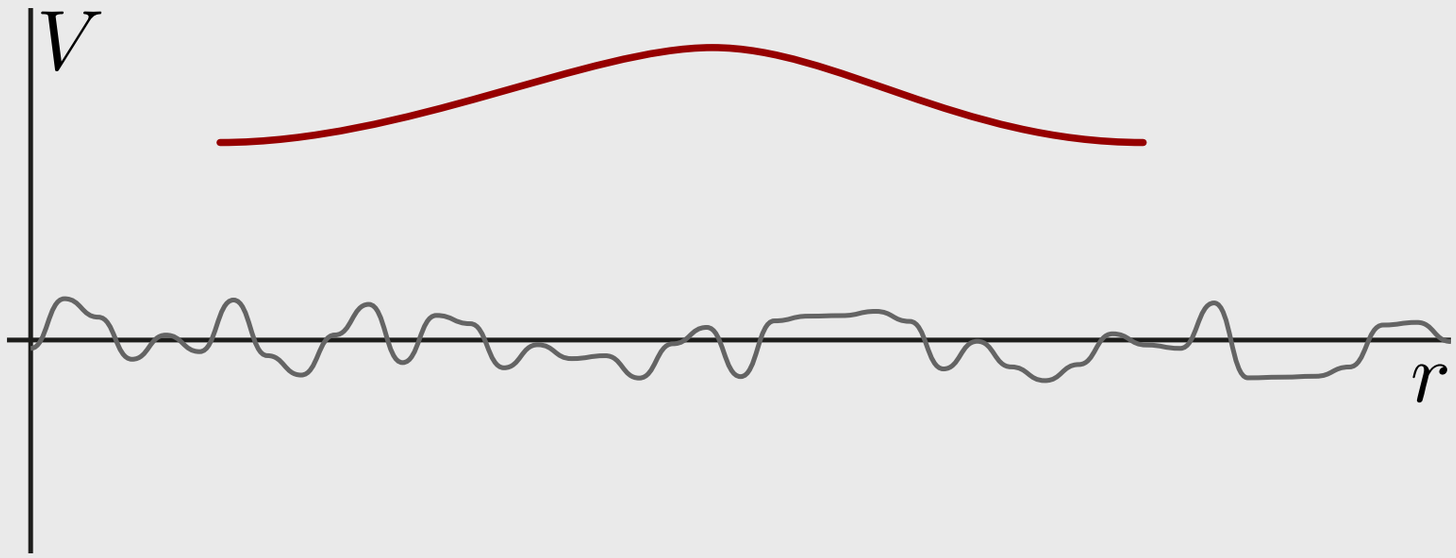


Topological insulators

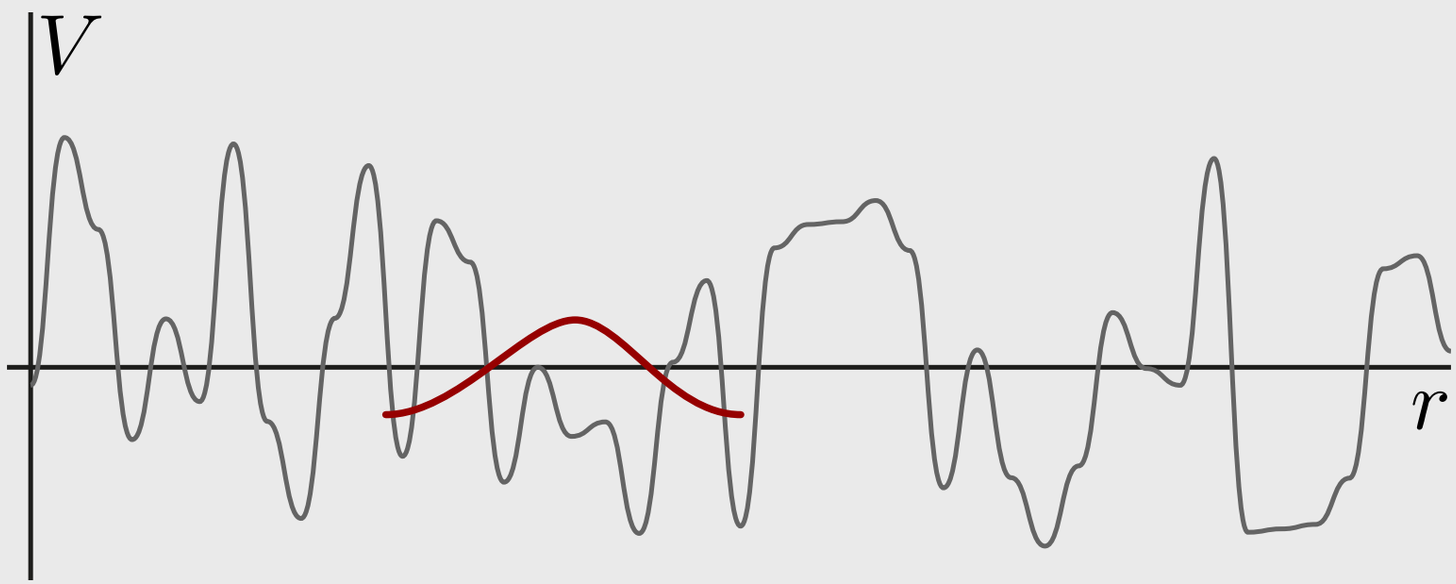
Anderson insulators

Anderson localization

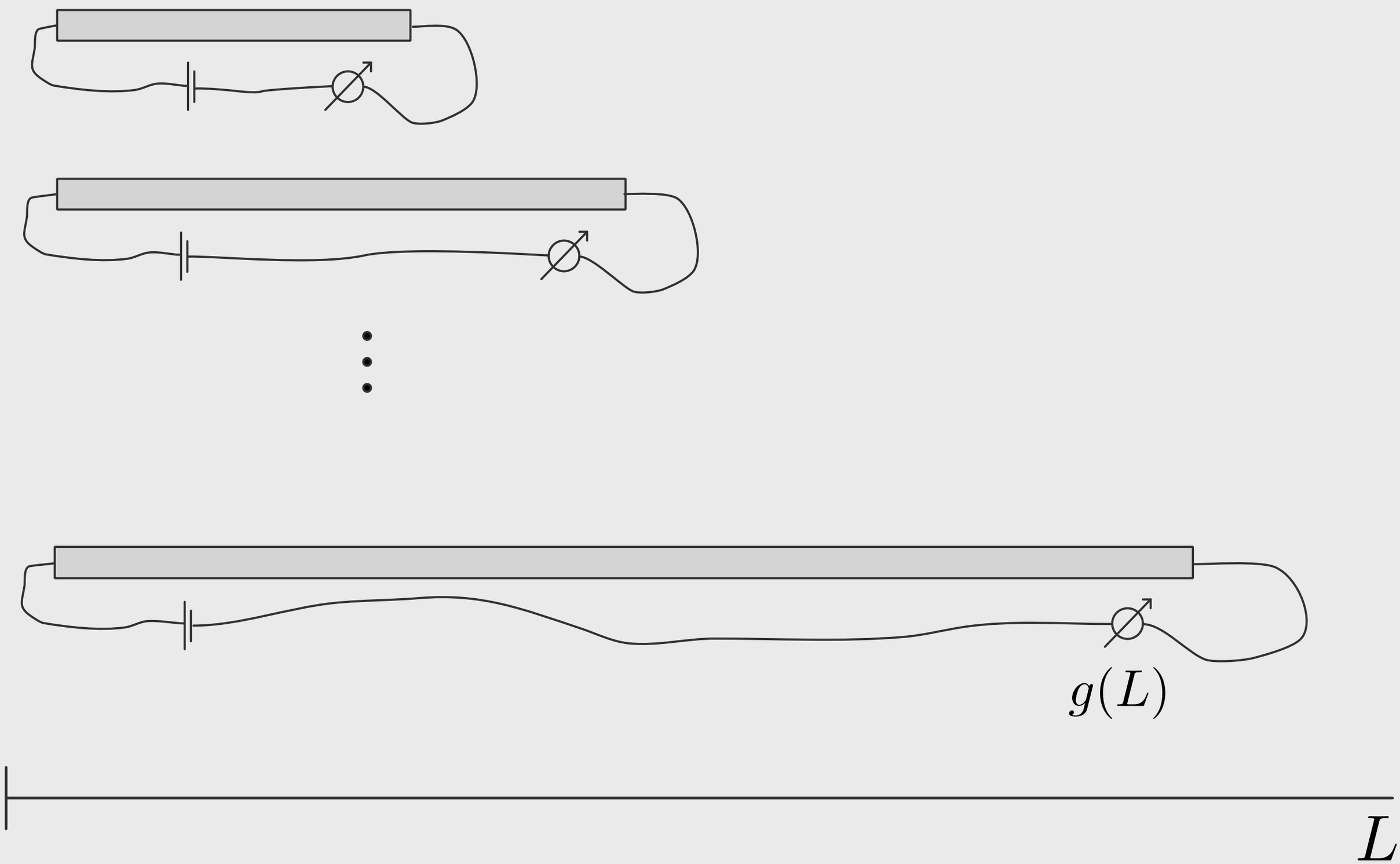
$d < 3$



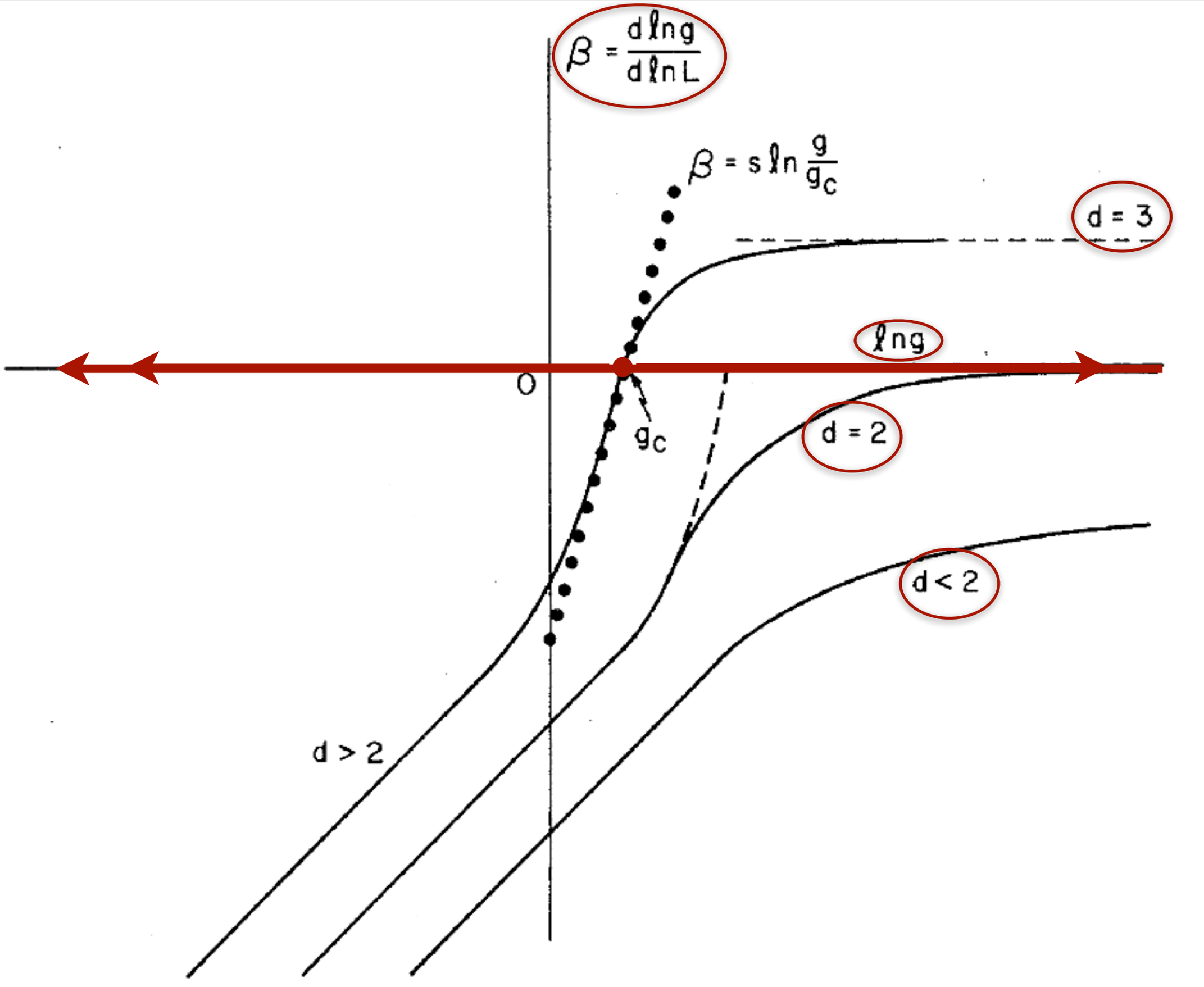
$d \geq 3$



one parameter scaling: idea



one-parameter scaling: pioneering theory



Abrahams et al. 79

Physics of Anderson insulator encoded in scaling of *g(L)*

disorder

Anderson insulator

Topological Anderson insulator

Topological insulator

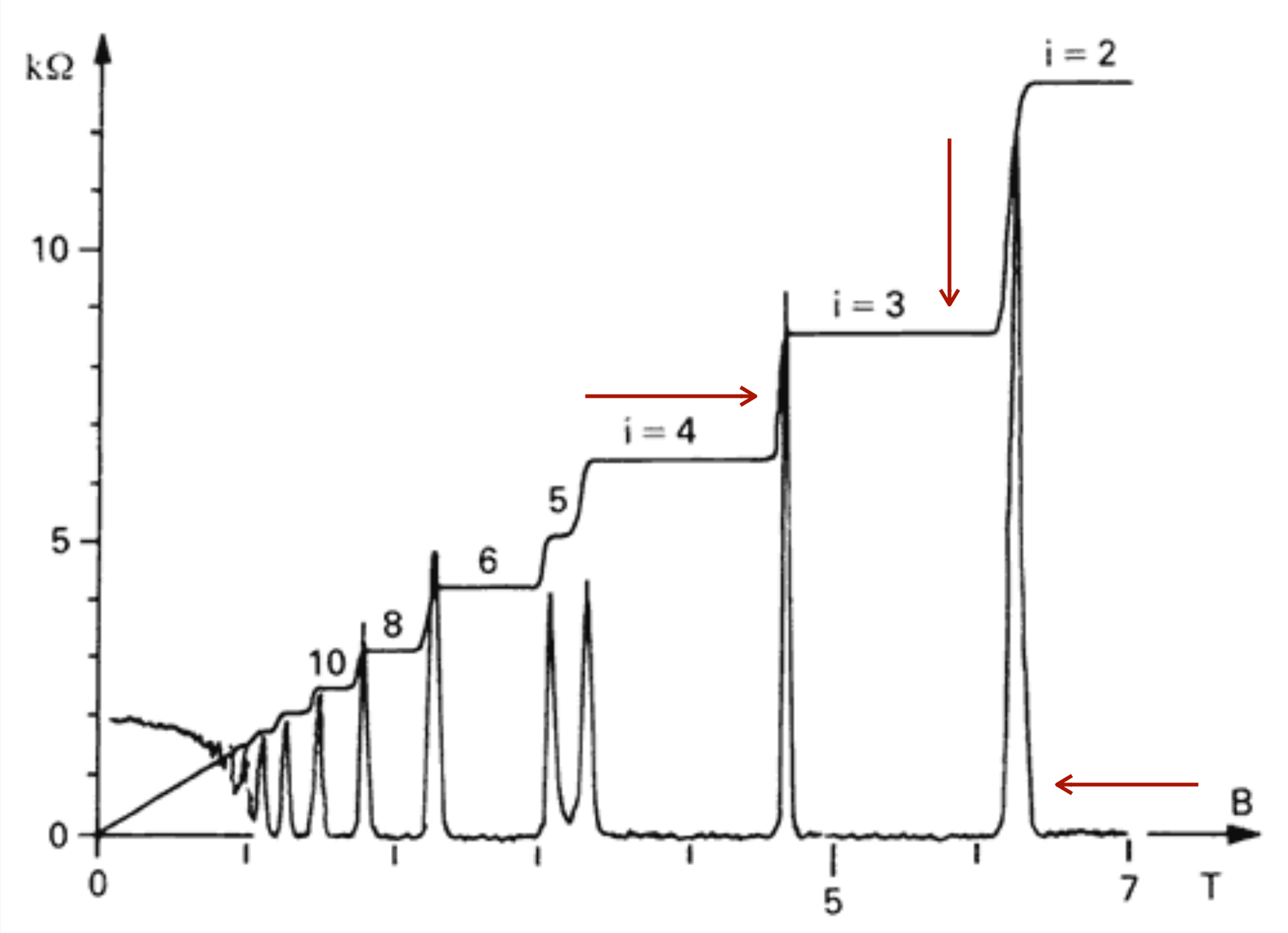
topology

case study: IQH

disordered topological matter — time line

quantum Hall effect (class A)

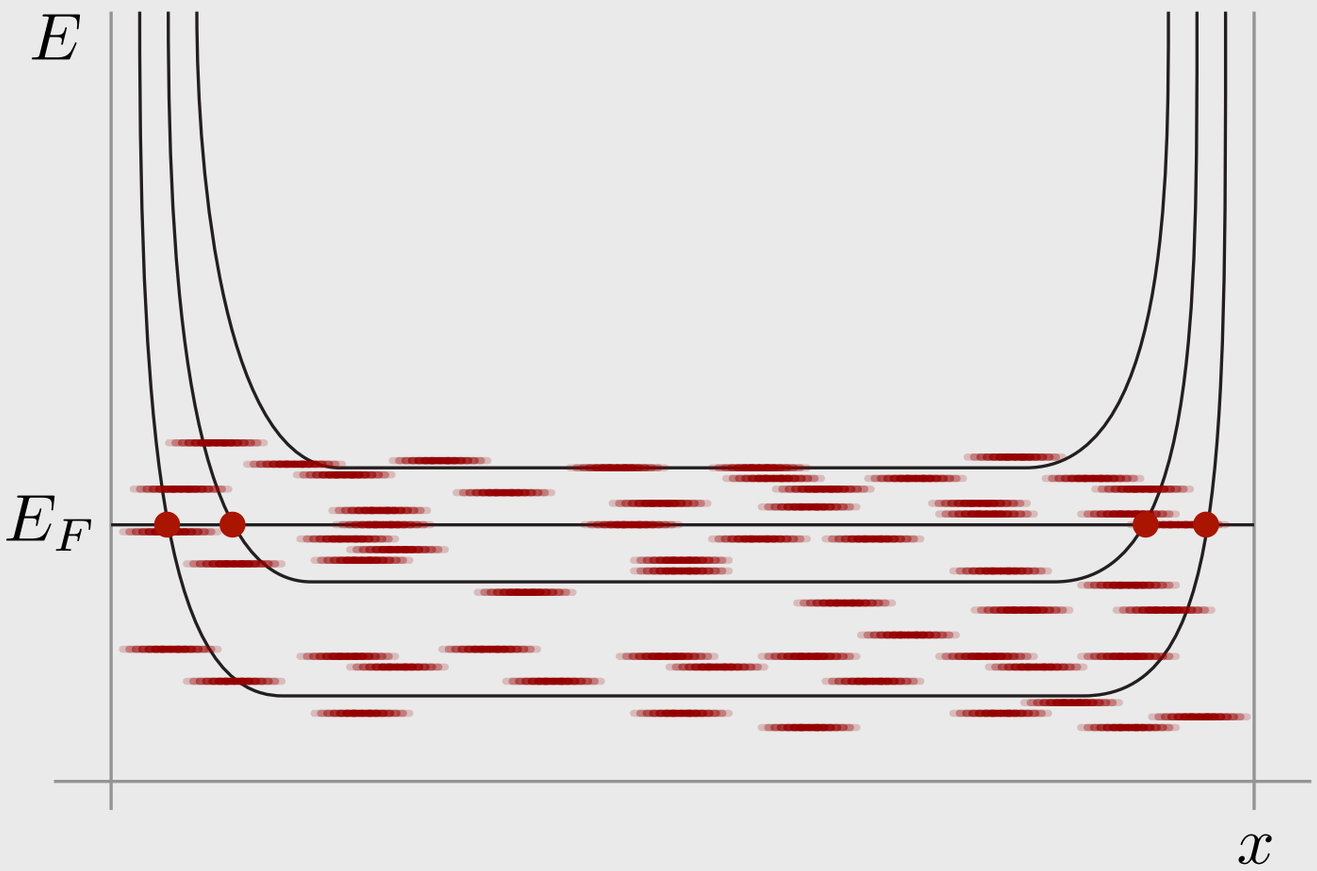
quantum Hall effect

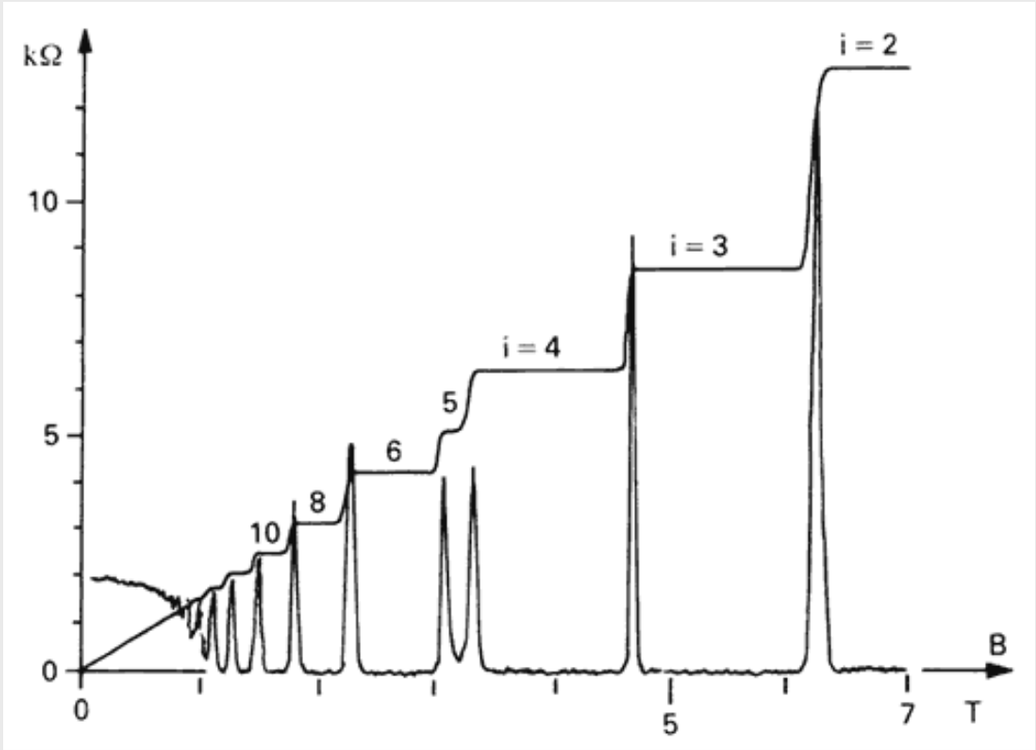


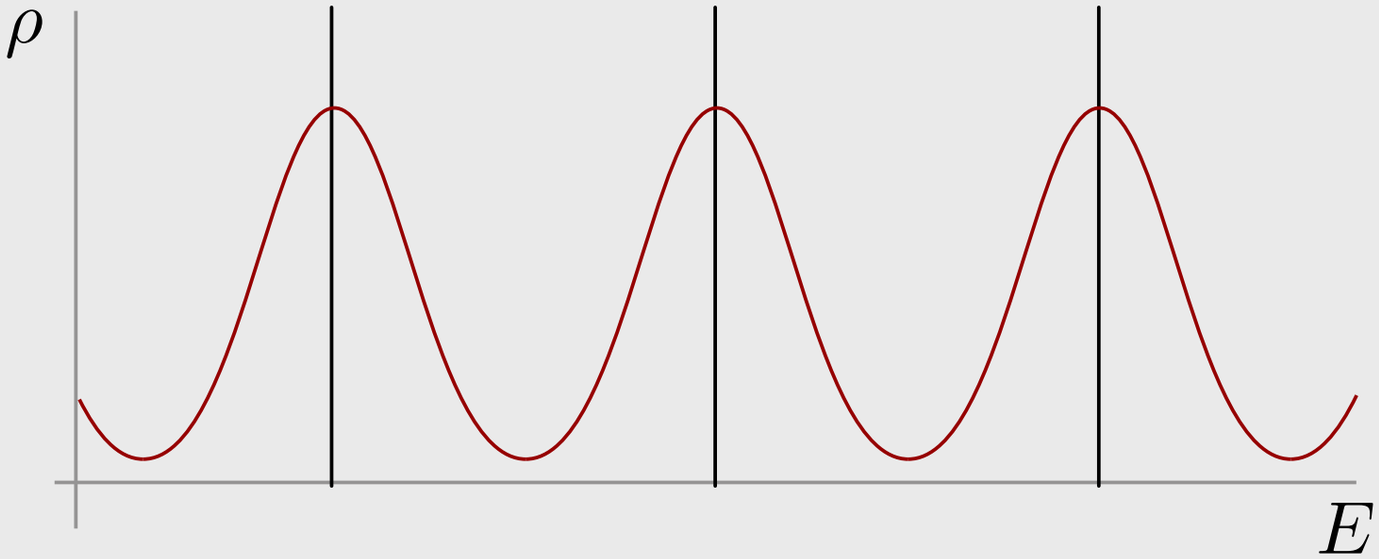
1998 Nobel prize press release

▷ smooth profiles of σ_{xx}, σ_{xy} becoming more singular upon lowering temperature

quantum Hall effect







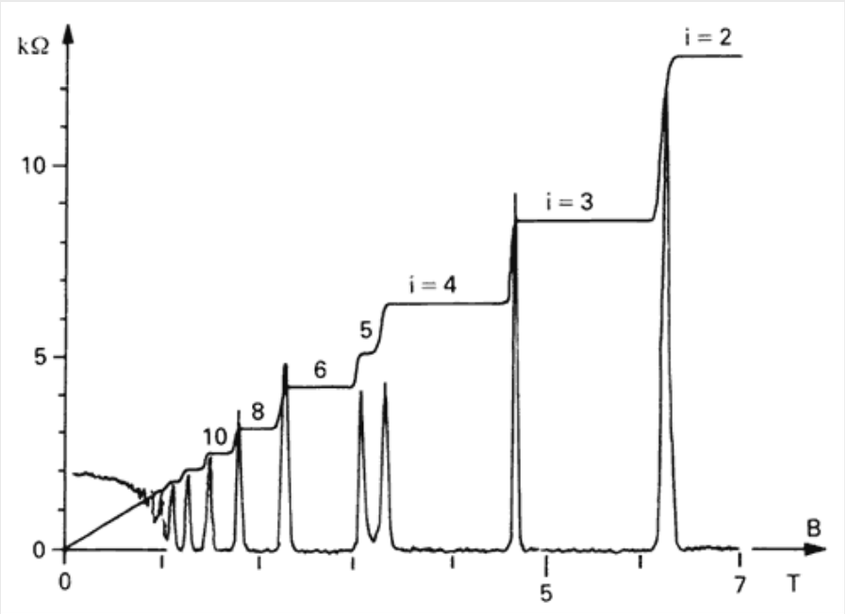
quantum Hall continued

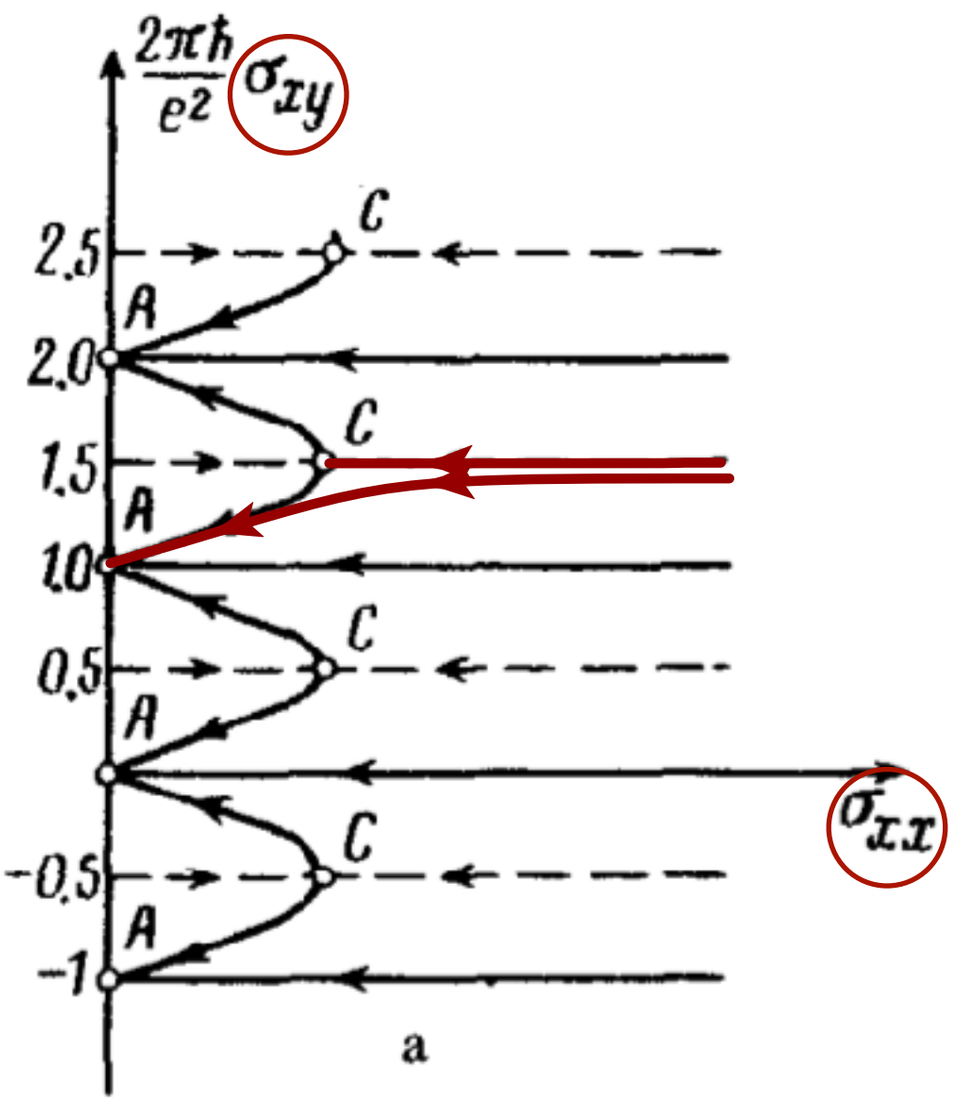
- system described by two scaling parameters

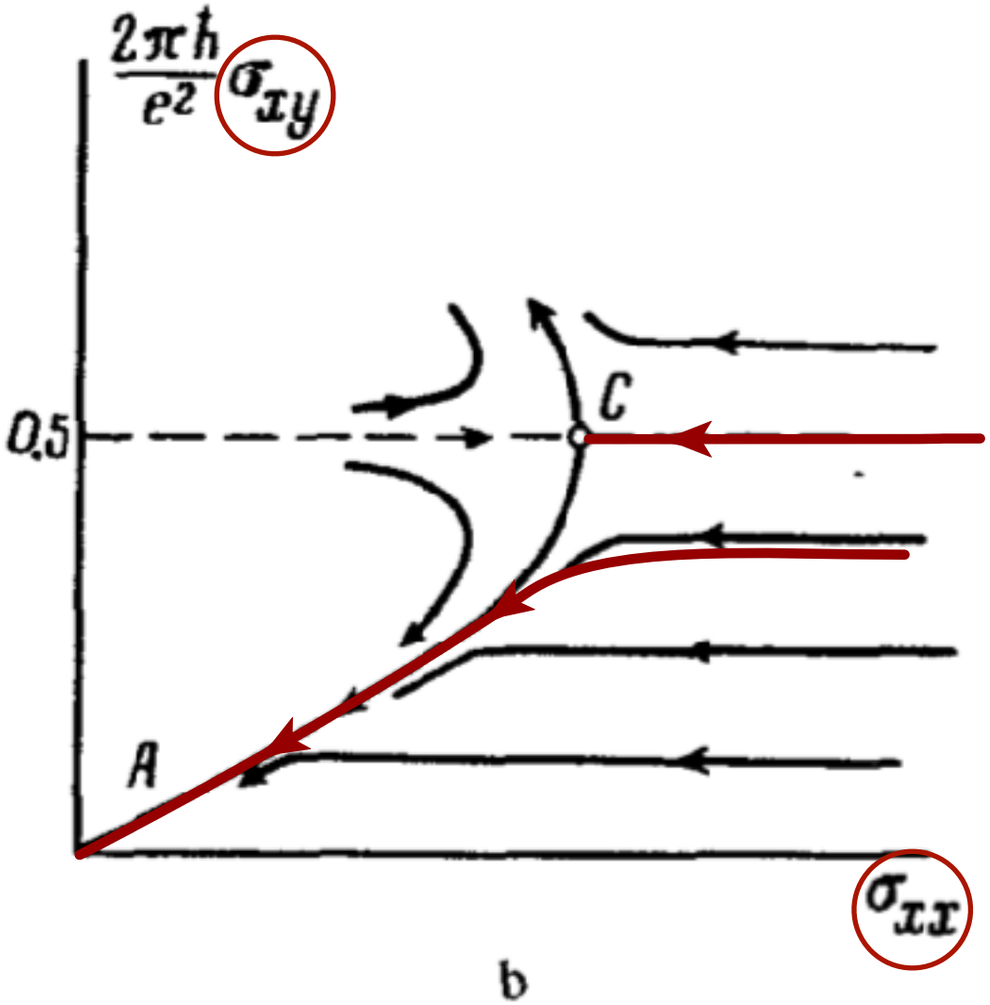
  σ_{xx} : average longitudinal transport coefficient

  σ_{xy} : average topological index

- two parameter criticality







Khmelnitskii, 1983

80s

quantum Hall

90s

symmetry classes

symmetries in QM

unitary symmetries

rotational, translational, inversion, point group, …

anti-unitary symmetries

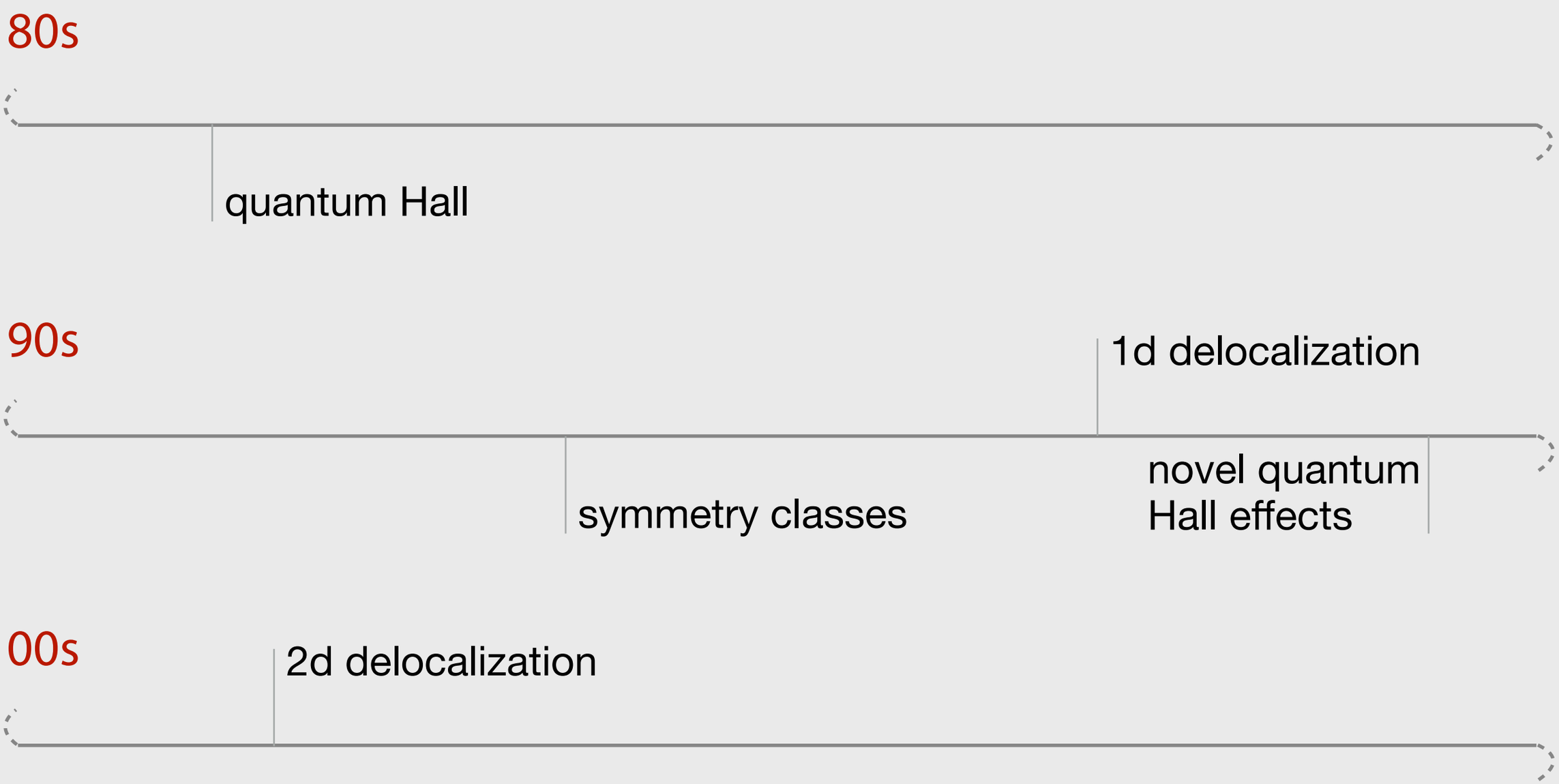
time reversal, particle hole, chiral symmetries, …

more robust

10 fundamental symmetry classes, A, AI, AII, AIII, BDI, D, C, CI, CII, DIII

organizing principle for the classification of topological matter (later)

disordered topological matter — time line



1d delocalization phenomena

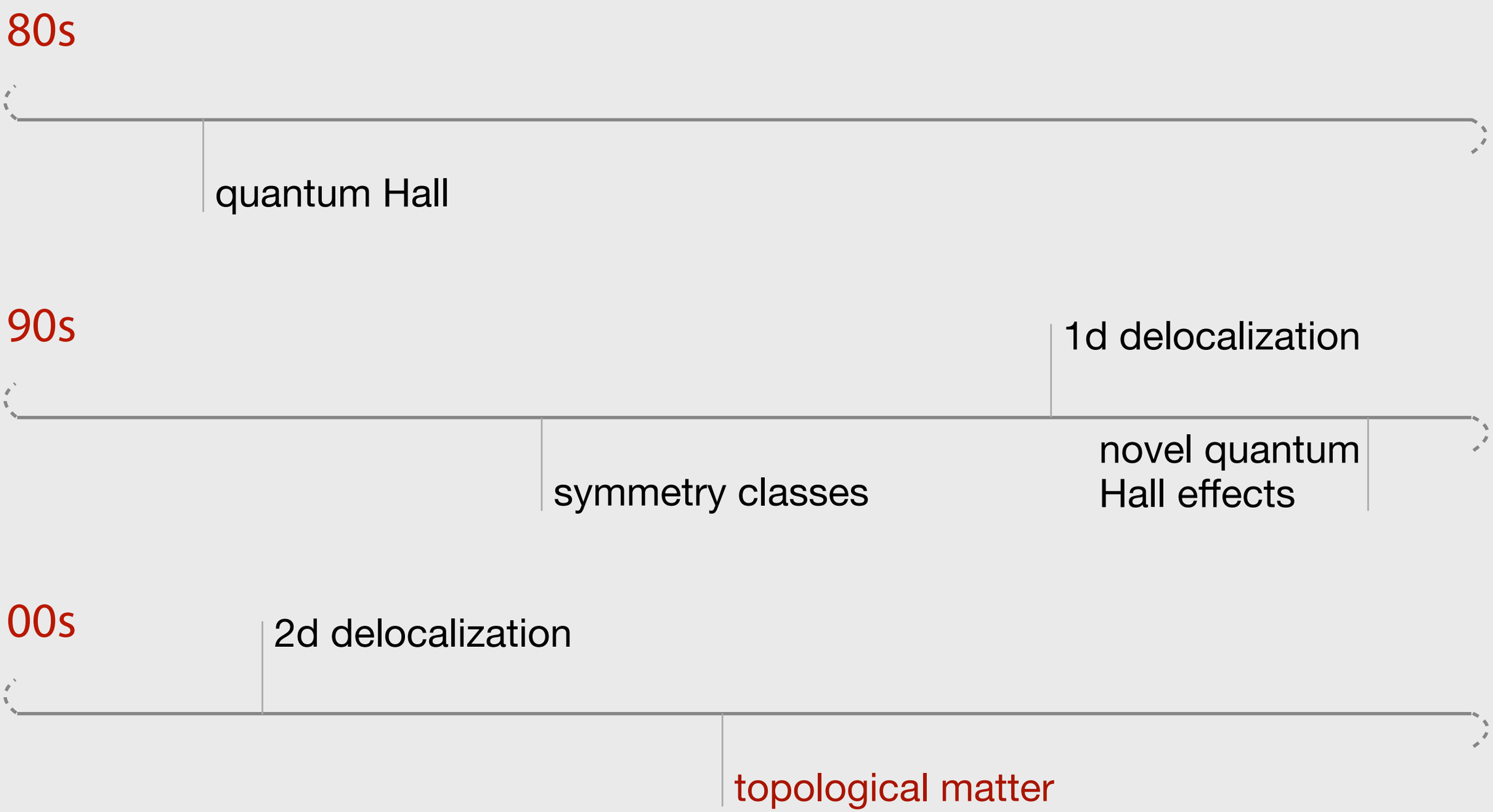
- delocalization in quasi-one dimensional geometries
  - 1998 AIII quantum wire (Brouwer, Mudry, Simons, AA)
  - 1999 D quantum wire (Brouwer, Mudry, Furusaki, Zirnbauer, Serban)
  - 2004 AIII, D, BDI, DIII, CII (Read, Gruzberg, Vishveshwara)

Unconventional criticality in 2d

- 2001 Class C spin quantum Hall effect (Chalker et al.)
- 2001 Class D quantum criticality (Fisher et al., Read et al.)

Universal interpretation: topological insulators at quantum critical point

disordered topological matter — time line



complex case:

Cartan\d	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$

IQH

SSH

real case:

Cartan\d	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
AI	$\mathbb{Z}$	0	[illegible]	[illegible]	[illegible]	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
CII	0	$2\mathbb{Z}$	[illegible]	[illegible]	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$

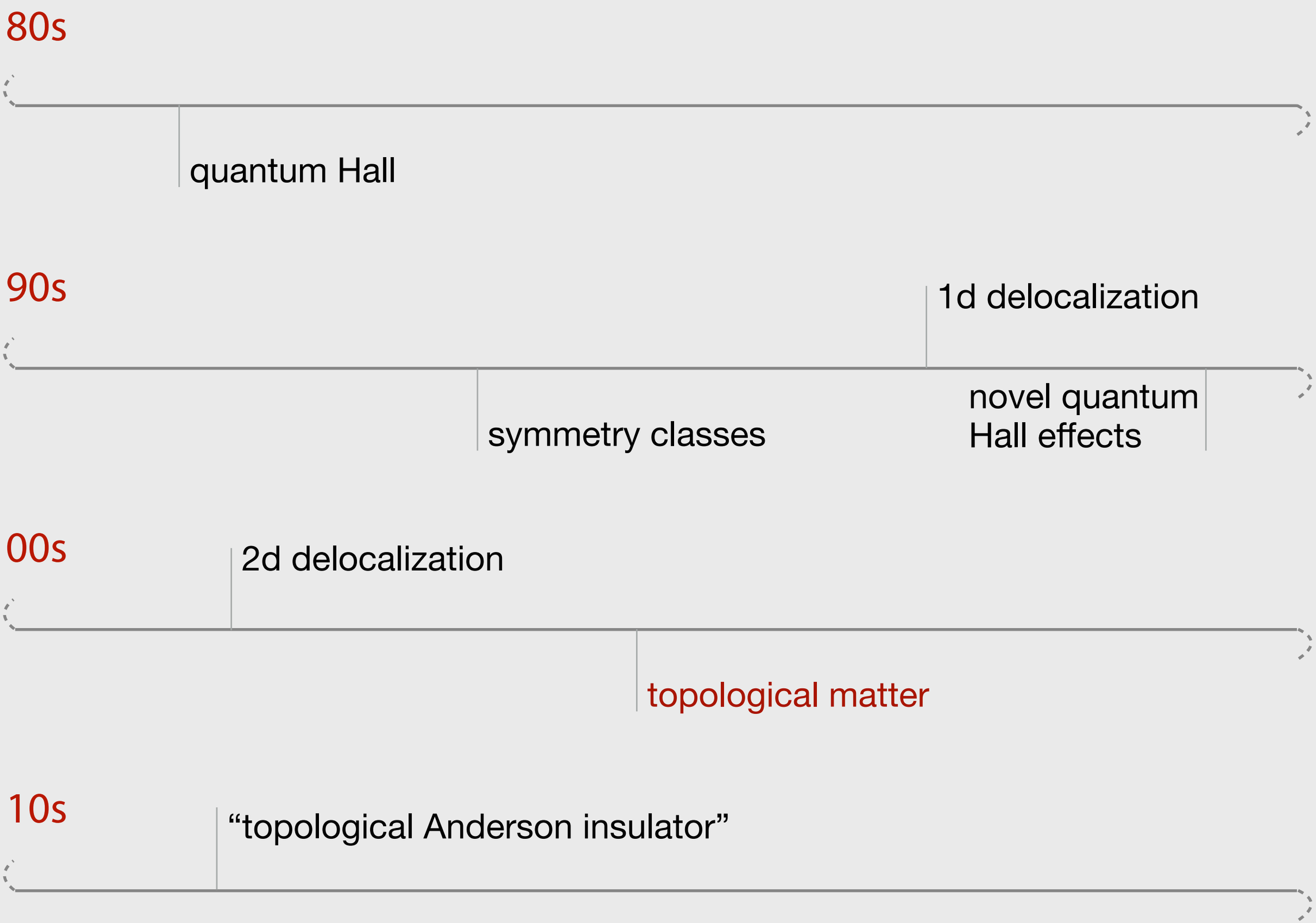
'Kitaev chain'

'Majorana quantum wire'

'topolgical insulator (HgTe)'

spin QH

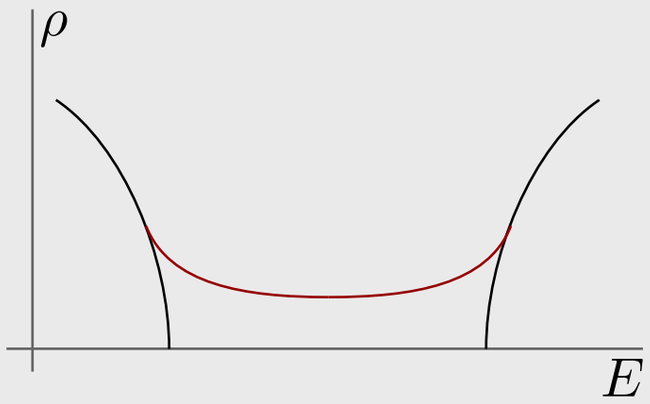
disordered topological matter — time line

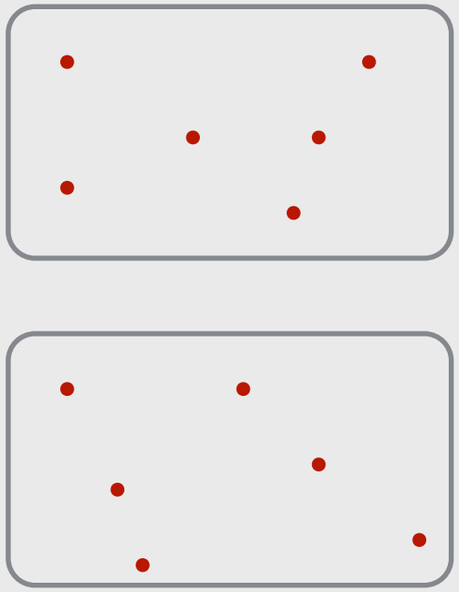


disorder vs. topology

Addition of disorder to a clean band insulator …

- ▷ invalidate k-theory of band gaps/topological indices
- ▷ destroys band gaps and
- ▷ turns insulator into nominal metal (-> Anderson ins.)
- ▷ renders topological indices statistically distributed





$n = 2$

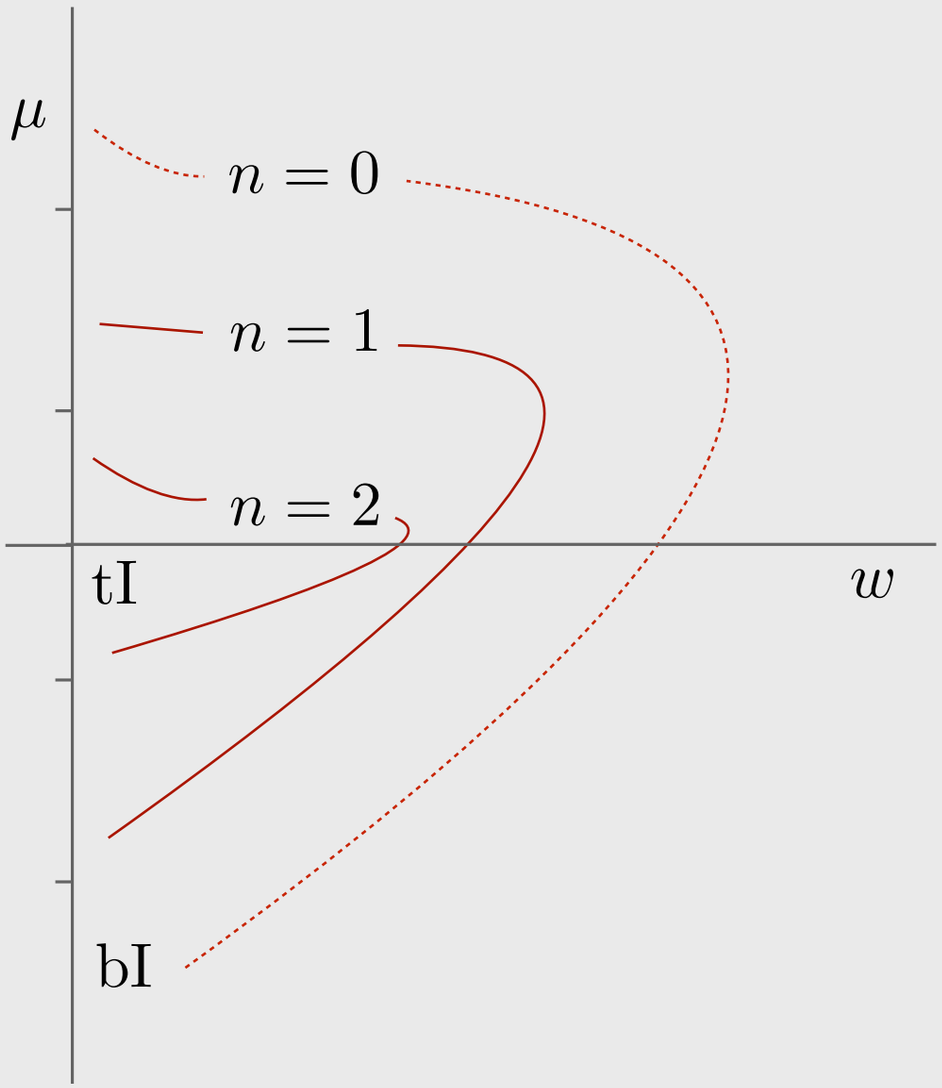
$n = 3$

	clean	disordered
k-theory	+	
band gap	+	
index	+	(+)

$\langle \cdots \rangle$

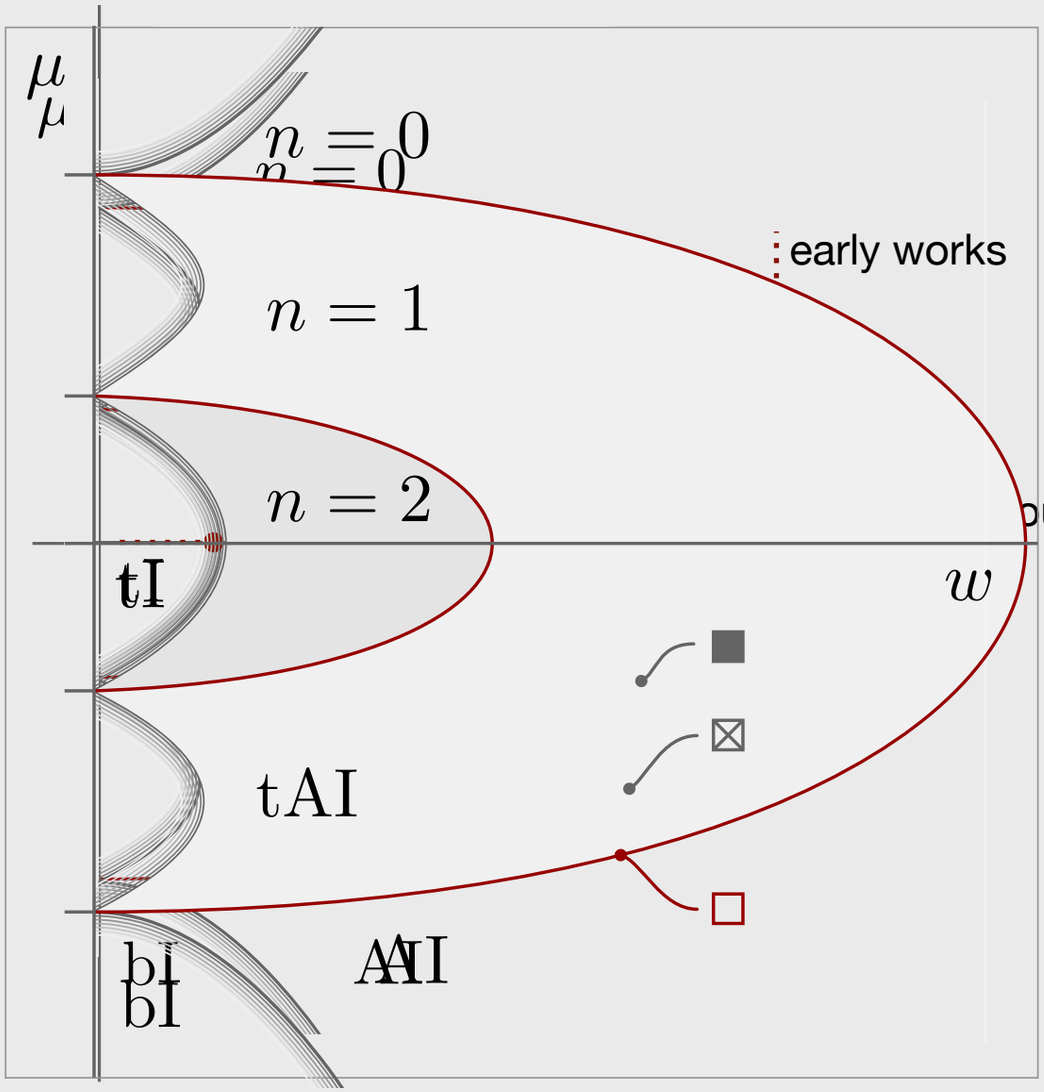
$\langle n \rangle \equiv \chi = ?$

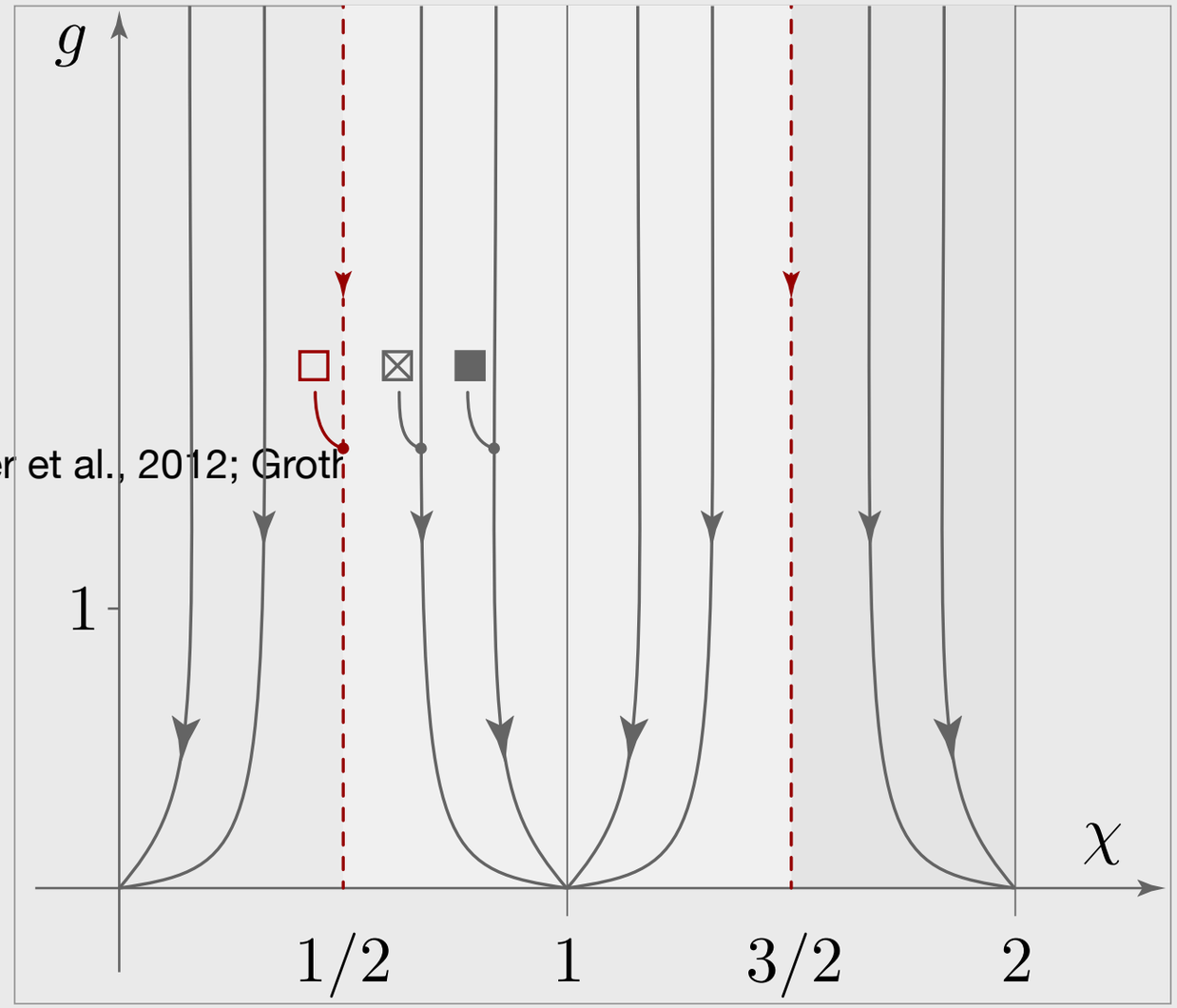
topological Anderson insulator



topological Anderson insulator

cf. Motrunich, Damle and Huse 2001, Groth et al. 2010





ouwer et al., 2012; Groth

critical theory ?

Universal theory of disordered TI — checklist

complex case:

Cartan\d	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$

IQH

SSH

real case:

Cartan\d	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
AI	$\mathbb{Z}$	0	[illegible]	[illegible]	[illegible]	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$\mathbb{Z}_2$	$\cdots$
CII	0	$2\mathbb{Z}$	[illegible]	[illegible]	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$

'Kitaev chain'

'Majorana quantum wire'

'topolgical insulator (HgTe)'

spin QH

1 and 2 dimensional topological insulators

	A	AIII	AI	BDI	D	DIII	AII	CII	C	CI
1										
2										

$\mathbb{Z}$

$\mathbb{Z}_2$

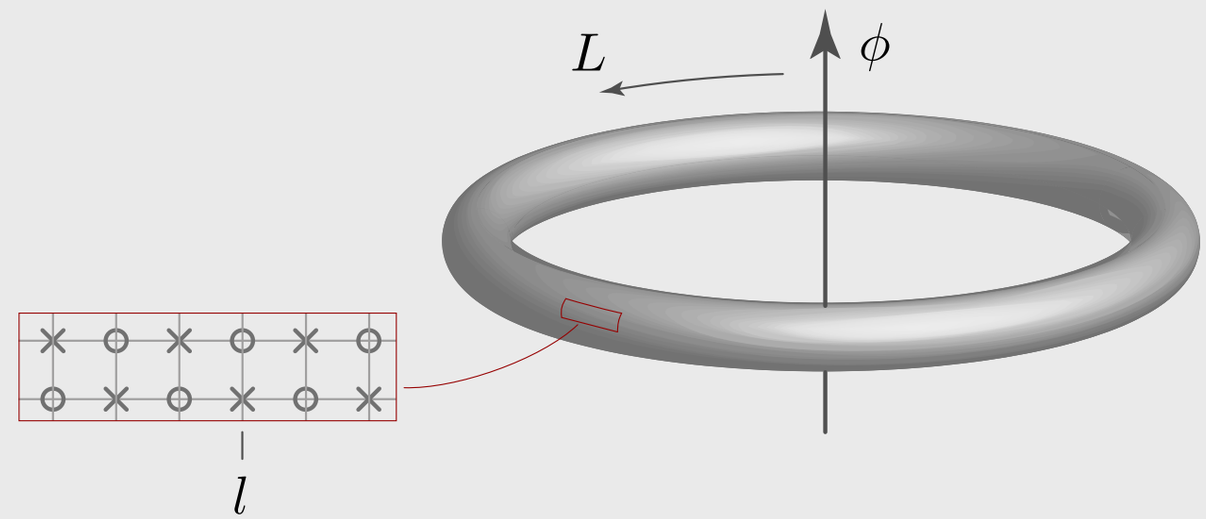
1d class AIII: SSH chain

generalized definition of winding number

$$n = -\frac{i}{2\pi}\int_0^{2\pi} dk\, \mathrm{tr}\,(\hat{Q}^{-1}\partial_k \hat{Q})$$

▷ generalized definition

$$|\circ_l\rangle \to e^{+i\frac{\phi l}{L}}\,|\circ_l\rangle$$
$$|\times_l\rangle \to e^{-i\frac{\phi l}{L}}\,|\times_l\rangle$$

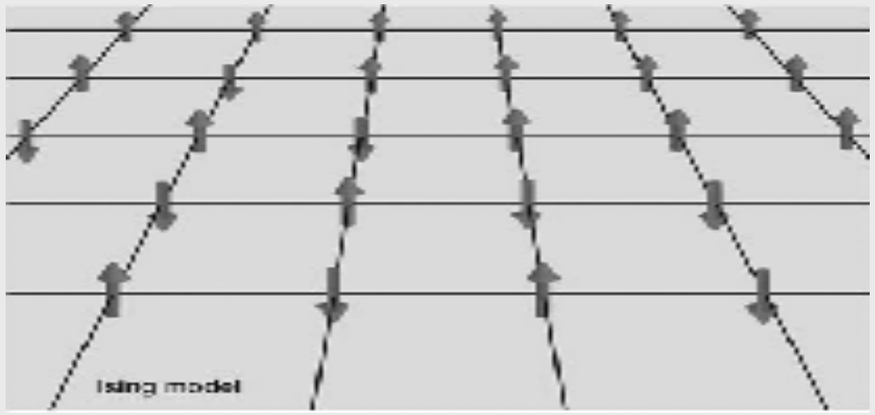


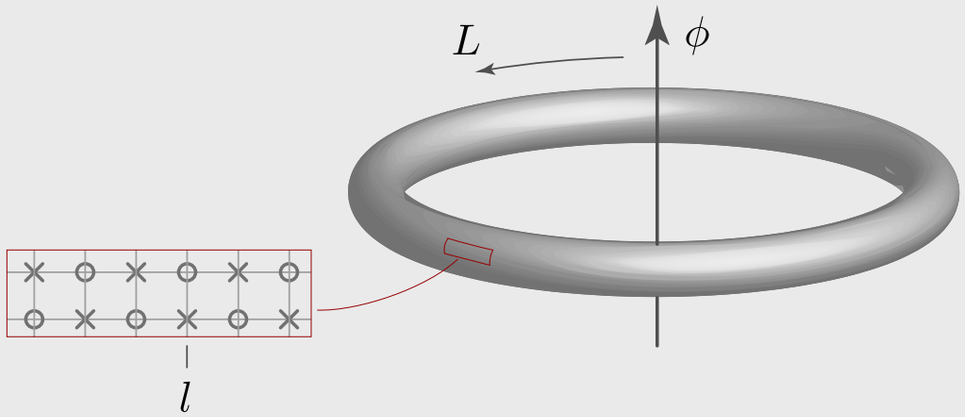
$$n \equiv \chi = -\frac{1}{2\pi}\Big[\ln\,\det(\hat{G}(2\pi)) - \ln\det(\hat{G}(0))\Big]$$

$$\hat{G}(\phi) = (i0 - \hat{H}(\phi))^{-1}$$

▷ want to know (g, χ) of disordered system

microscopic system





critical phenomena

$$Z = \int \mathcal{D}\varphi \, e^{-S[\varphi]}$$

$$S[\varphi] = \int d^d r \left(\frac{1}{2} \partial\varphi^2 + \frac{m}{2}\varphi^2 + g\varphi^4 \right)$$

$$Z = \int \mathcal{D}T \, e^{-S[T]}$$

$$S[T] = \int d^d x \, \tilde{g} \, \mathrm{str}(\partial T \partial T^{-1}) + \tilde{\chi} \, S_{top}[T]$$

Ginzburg Landau theory

field integral

$$Z(\phi) = \int \mathcal{D}T \, \exp(-S[T])$$

$$S[T] = \int_0^L dx \left[\frac{\tilde{\xi}}{4} \, \mathrm{str}(\partial_x T \partial_x T^{-1}) + \tilde{\chi} \, \mathrm{str}(T^{-1} \partial_x T) \right]$$

AA & Merkt, 2001

- matrix fields $T = U \begin{pmatrix} e^{y_1} & \\ & e^{iy_0} \end{pmatrix} U^{-1}$
- boundary conditions $T(L) = T(0) \begin{pmatrix} e^{\phi_1} & \\ & e^{i\phi_0} \end{pmatrix}$
- $(\tilde{\xi}, \tilde{g})$ bare (SCBA) values of loc. length and topological parameter, resp.

$$\tilde{\xi} = Nl, \qquad \tilde{\chi} = -\frac{i}{2} \mathrm{tr}(\hat{G}^+ \hat{v})$$

- good picture: path integral of quantum point particle in time L.

diffusive systems

$$Z(\phi) = \int \mathcal{D}T \, \exp(-S[T])$$

$$S[T] = \int_0^L dx \left[\frac{\tilde{\xi}}{4} \, \mathrm{str}(\partial_x T \partial_x T^{-1}) + \tilde{\chi} \, \mathrm{str}(T^{-1} \partial_x T) \right]$$

- for $L < \tilde{\xi}$: $\quad T(x) = \mathrm{diag}(e^{\phi_1 x/L}, e^{i\phi_0 x/L})$,

$$S[T] = \frac{\tilde{\xi}}{L}(\phi_1^2 + \phi_2^2) + \tilde{\xi}(\phi_1 + i\phi_0)$$

- diffusive conductance $g = \dfrac{\tilde{\xi}}{L}$

- non-integer average invariant $\chi = \tilde{\chi}$

Understanding the path integral

▷ focus on compact sector: $y_0 = \theta$

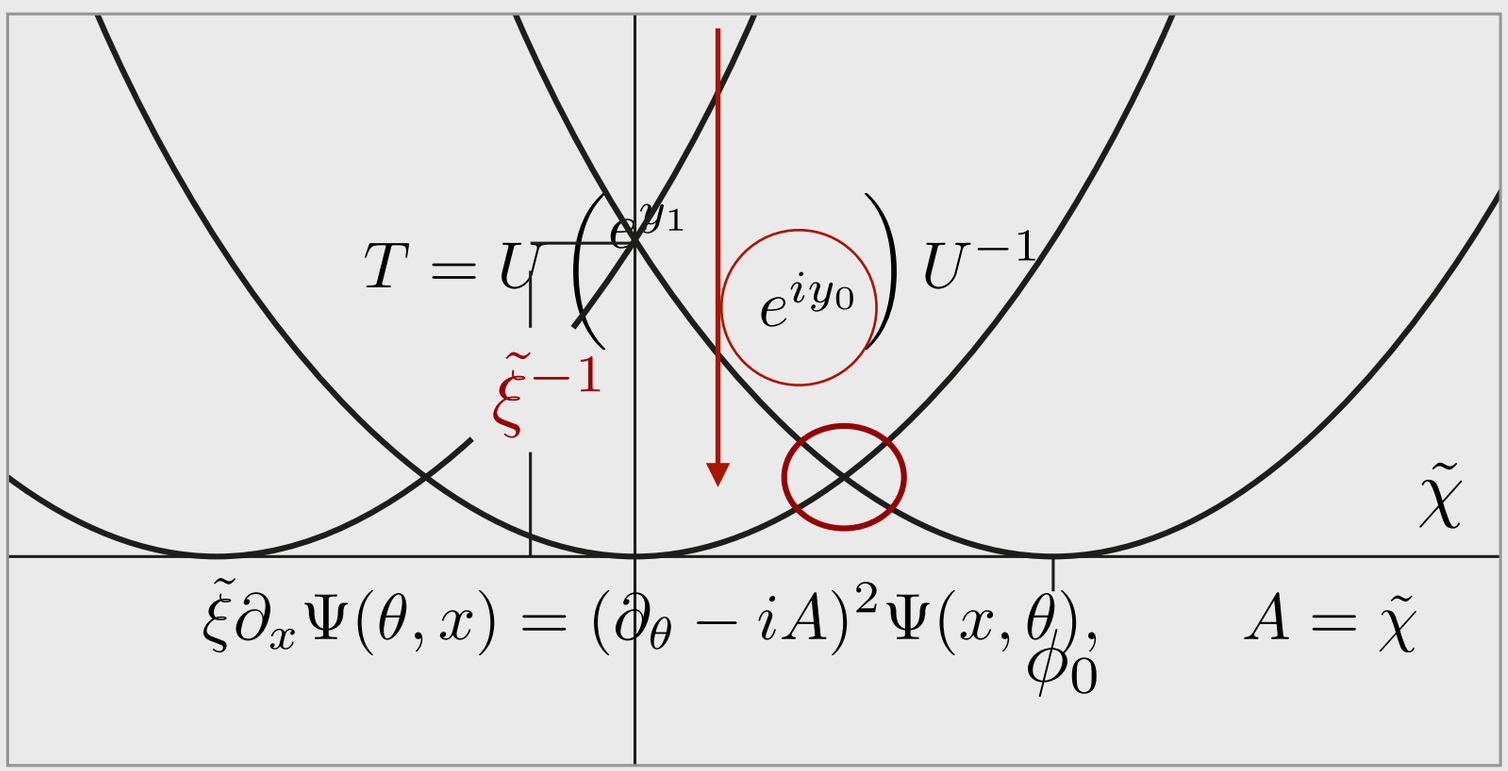
▷ theory reduces to QM on a ring with twisted boundary conditions and subject to magnetic flux $\tilde{\chi}$

$$Z(\phi) = \int \mathcal{D}T \, \exp(-S[T])$$

$$Z(\phi) = \int \mathcal{D}\theta \, \exp(-S[\theta])$$

$$S[T] = \int_0^L dx \left[\frac{\tilde{\xi}}{4} \mathrm{str}(\partial_x T \partial_x T^{-1}) + \tilde{\chi} \, \mathrm{str}(T^{-1} \partial_x T) \right]$$

$$S[\theta] = \int_0^L dx \left[\frac{\tilde{\xi}}{4} \partial_x \theta \partial_x \theta + i \tilde{\chi} \partial_x \theta \right]$$



transfer matrix solution of full problem

$$Z(\phi) = \int \mathcal{D}T \, \exp(-S[T])$$

$$\tilde{\xi}\, \partial_x \Psi(y,x) = \frac{1}{J(y)} (\partial_\alpha - iA_\alpha) J(y) (\partial_\alpha - iA_\alpha) \Psi(y,x),$$

$$J(y) = \sinh\left(\frac{1}{2}(y_1 - iy_0)\right)$$

$$S[T] = \int_0^L dx \left(\frac{1}{2}\left[\frac{\tilde{\xi}}{4} \mathrm{str}(\partial_x T \partial_x T^{-1}) + \tilde{\chi} A \mathrm{str}(T^{-1} \partial_x T) \right] \right)$$

$$A = \chi(\mathbf{r}, t)$$

- ▷ solvable problem:
  - ▷ eigenfunctions: $\Psi_l(y) = \sinh\left(\frac{1}{2}(y_1 - iy_0)\right) e^{il_\alpha y_\alpha} \qquad (l_0, l_1) \in (\mathbb{Z} + \frac{1}{2}, \mathbb{R})$
  - ▷ eigenvalues: $\epsilon_l = (l_0 - \tilde{\chi})^2 + (l_1 - i\tilde{\chi})^2$
  - ▷ solution by spectral sum:

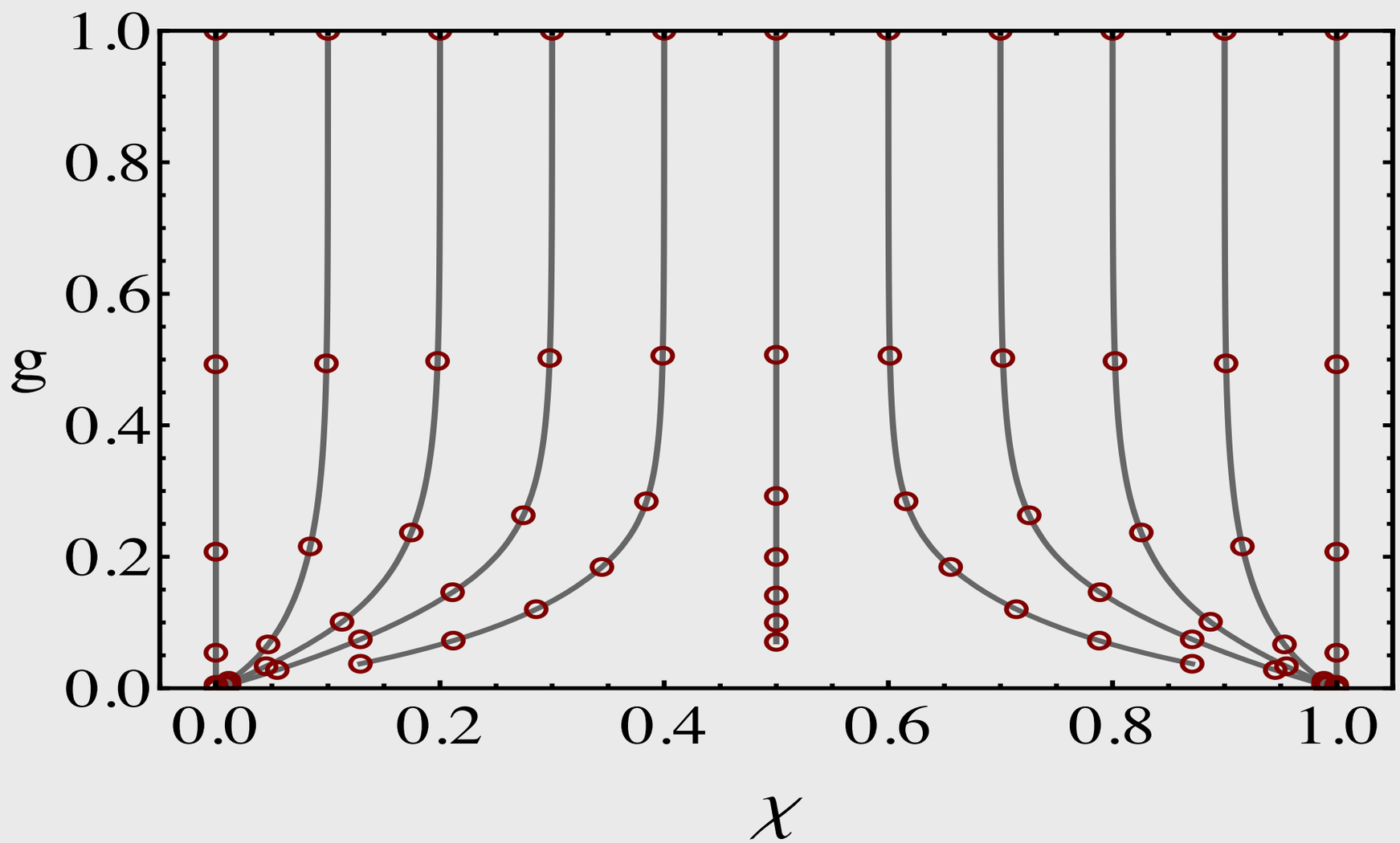
$$\Psi(\phi, L) = 1 + \frac{1}{\pi} \sum_{l_0 \in \mathbb{Z} + \frac{1}{2}} \int dl_1 \, \frac{\Psi_l(\phi)}{l_0 + il_1} \, e^{-\epsilon_l L/\tilde{\xi}}$$

results (AIII quantum wire)

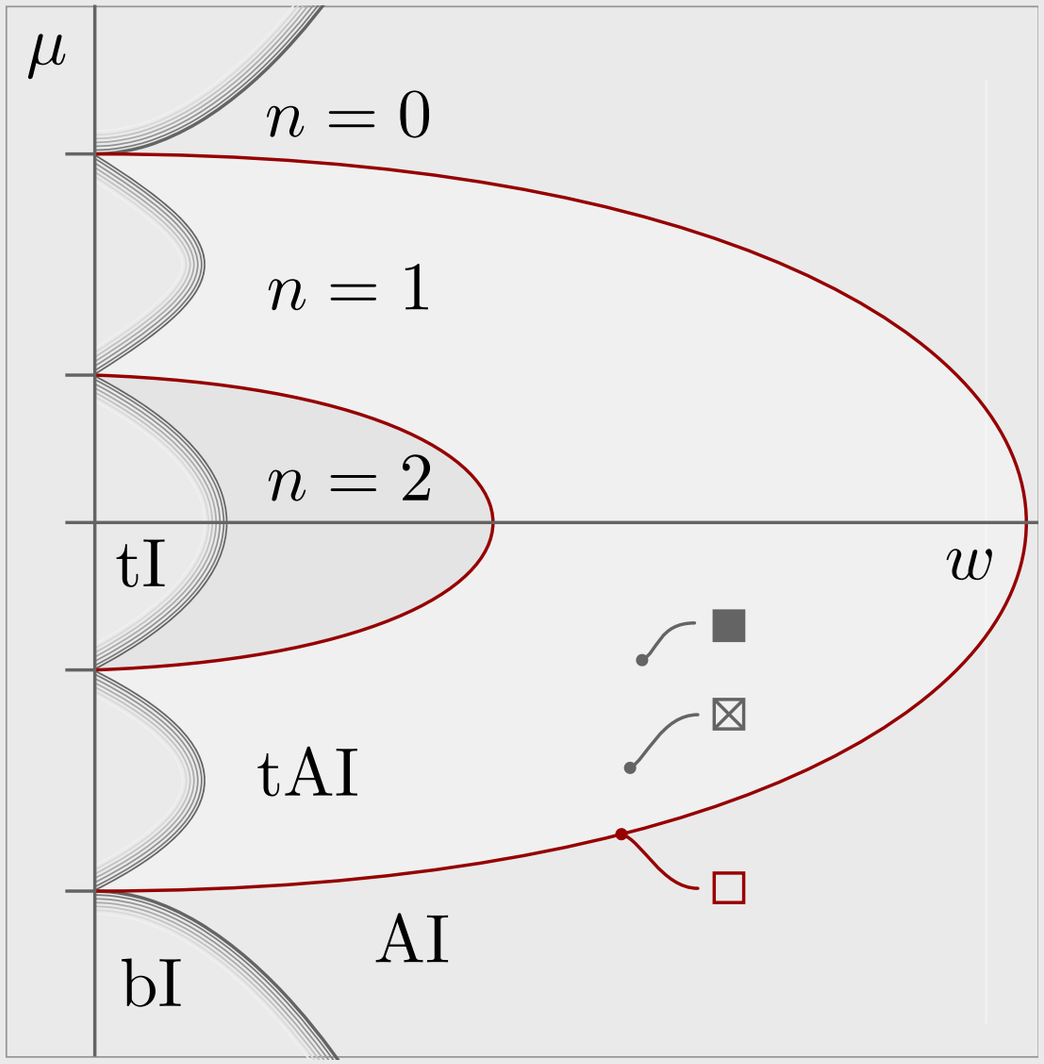
$$g = \sqrt{\frac{\tilde{\xi}}{\pi L}} \sum_{l_0 \in \mathbb{Z}+1/2} e^{-(l_0-\tilde{\chi})^2 L/\tilde{\xi}},$$

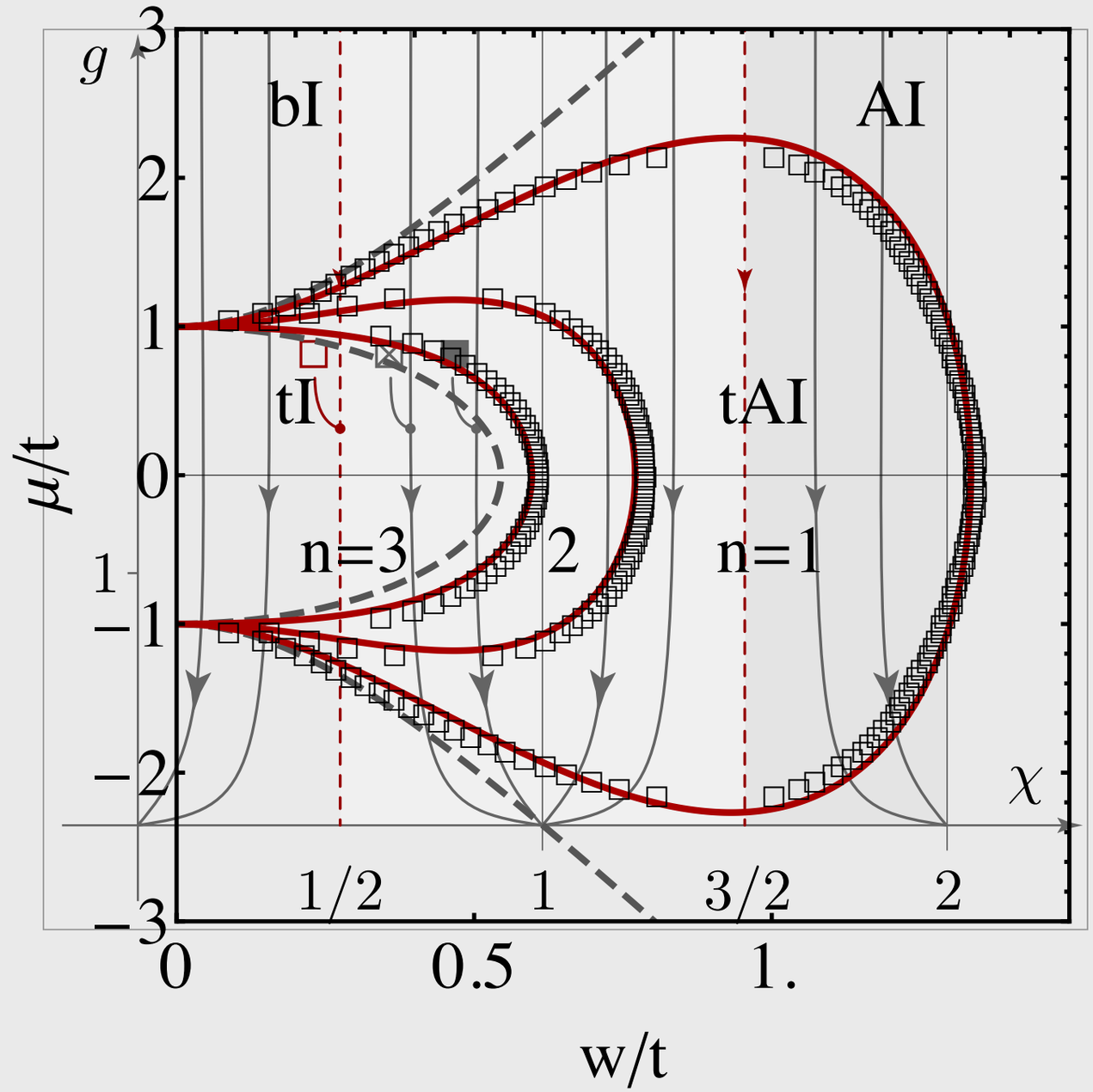
$$\chi = n - \frac{1}{4} \sum_{l_0 \in \mathbb{Z}+1/2} \left[\mathrm{erf}\left(\sqrt{\frac{L}{\tilde{\xi}}} \left(l_0 - \delta\tilde{\chi}\right) \right) - \left(\delta\tilde{\chi} \leftrightarrow -\delta\tilde{\chi}\right) \right],$$

where $\chi = n + \tilde{\chi}$



phase diagram





▷ phase boundaries: lines of half integer bare topological index

▷ flow describes stabilization of self-averaging topological phase/boundary state generation

1d class D
(‘Majorana quantum wire’)

Class D system

- superconductor lacking spin-rotation and time reversal invariance.

$$\boxed{\hat{H} = -\hat{H}^T}$$

- physical realization: proximity coupled spin-orbit semiconductor quantum wires

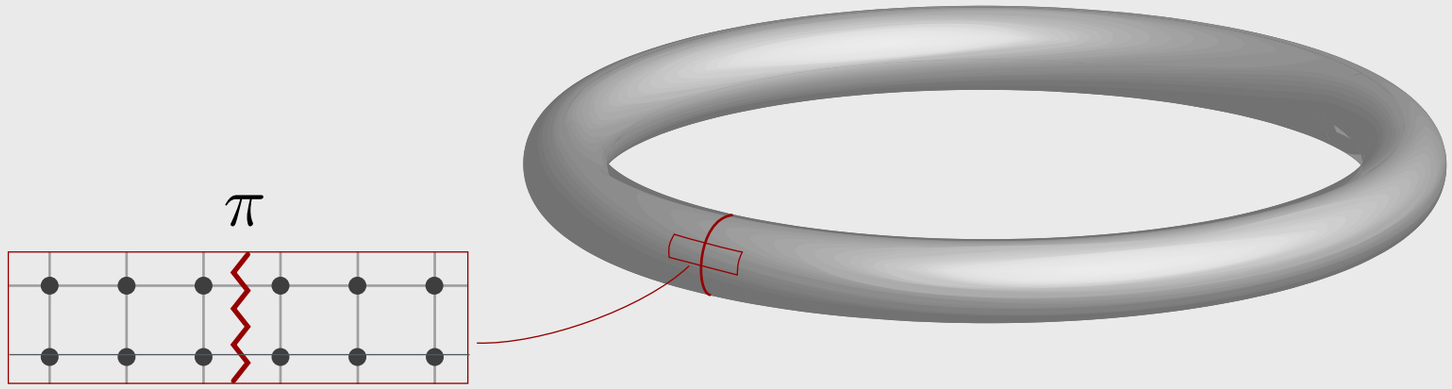
- momentum space definition of topological invariant

1.) $\hat{G} \equiv (i0 - \hat{H})^{-1}$

2.) $\hat{H}, \hat{G} \longrightarrow H(k), G(k) \qquad k \in [0, 2\pi[$

3.) $\chi \equiv \mathrm{sgn}\left(\frac{\mathrm{Pf}(G(\pi))}{\mathrm{Pf}(G(0))}\right) = \pm 1$

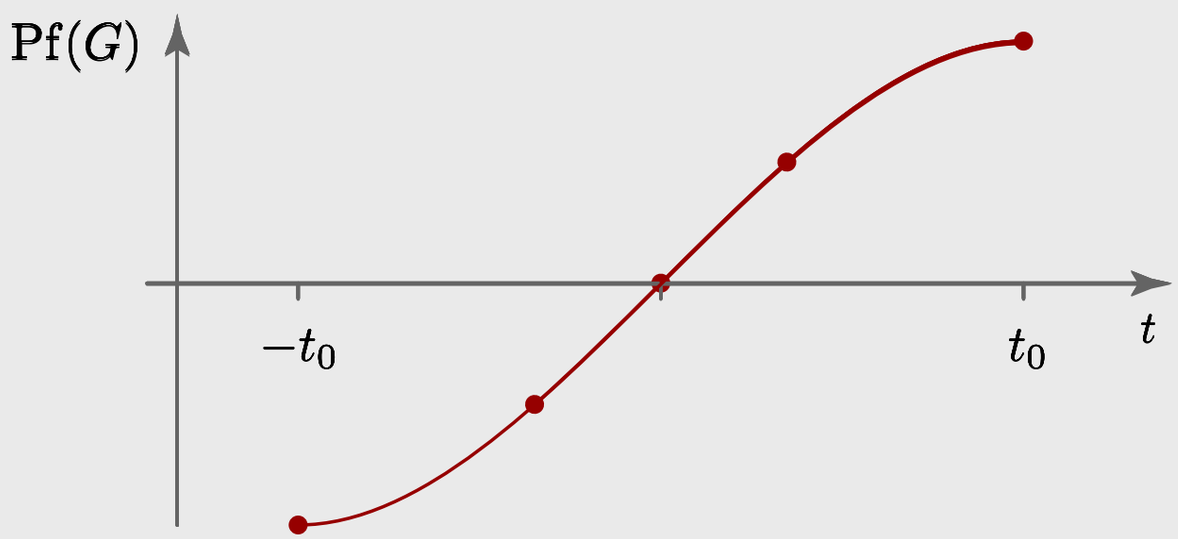
generalized definition of topological invariant



1.) $\hat{G}_0$: unperturbed Green function

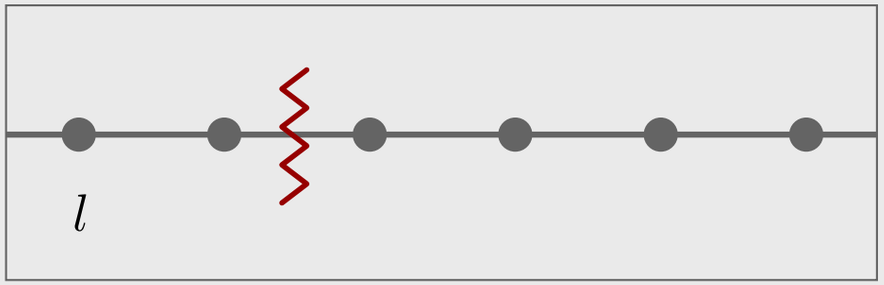
$\hat{G}_\pi$: Green function corresponding to one sign inverted column of hopping amplitudes

2.) $\chi \equiv \mathrm{sgn}\left(\frac{\mathrm{Pf}(G_0)}{\mathrm{Pf}(G_\pi)}\right) = \pm 1$



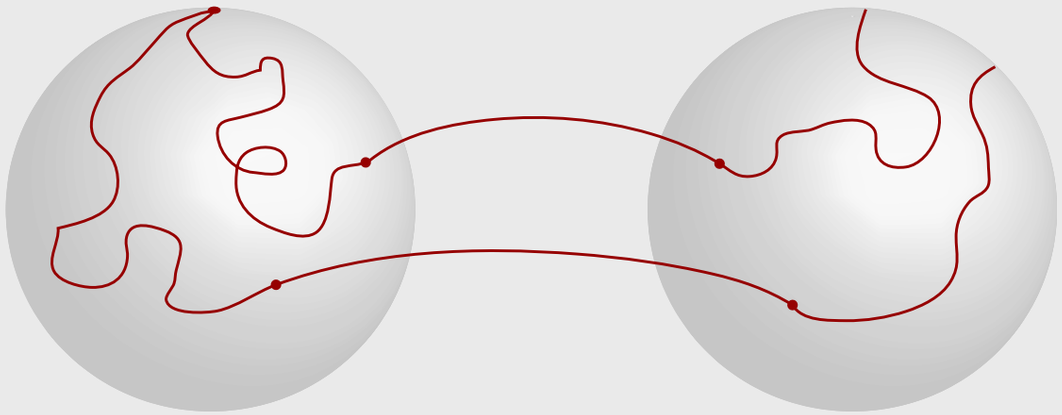
class D field theory

$$Z(\phi) = \int DQ \, \exp(-S_{\text{lattice}}[Q])$$



$$Q : [0, L] \simeq S^1 \longrightarrow (\text{CI}|\text{DIII})\big|_1$$
$$x \longmapsto Q(x)$$

▷ fermion-fermion sector: $Q_{\text{ff}} = \mathbb{Z}_2$



transfer matrix equation

$$\partial_x \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix} = \frac{1}{\tilde{\xi}} \begin{pmatrix} \frac{1}{2} B^\dagger B & \tilde{\chi} B^\dagger \\ -\tilde{\chi} B & \frac{1}{2} B B^\dagger \end{pmatrix} \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix}$$

$$\Psi = \Psi(\phi)$$

$$B = -\partial_\phi + W(\phi) \qquad W(\phi) = \frac{1}{2}(\coth(\phi) - \tanh(\phi))$$

- cf. SUSY quantum mechanics
- at criticality ($\tilde{\chi} = 0$, or absence of kinks): delocalization

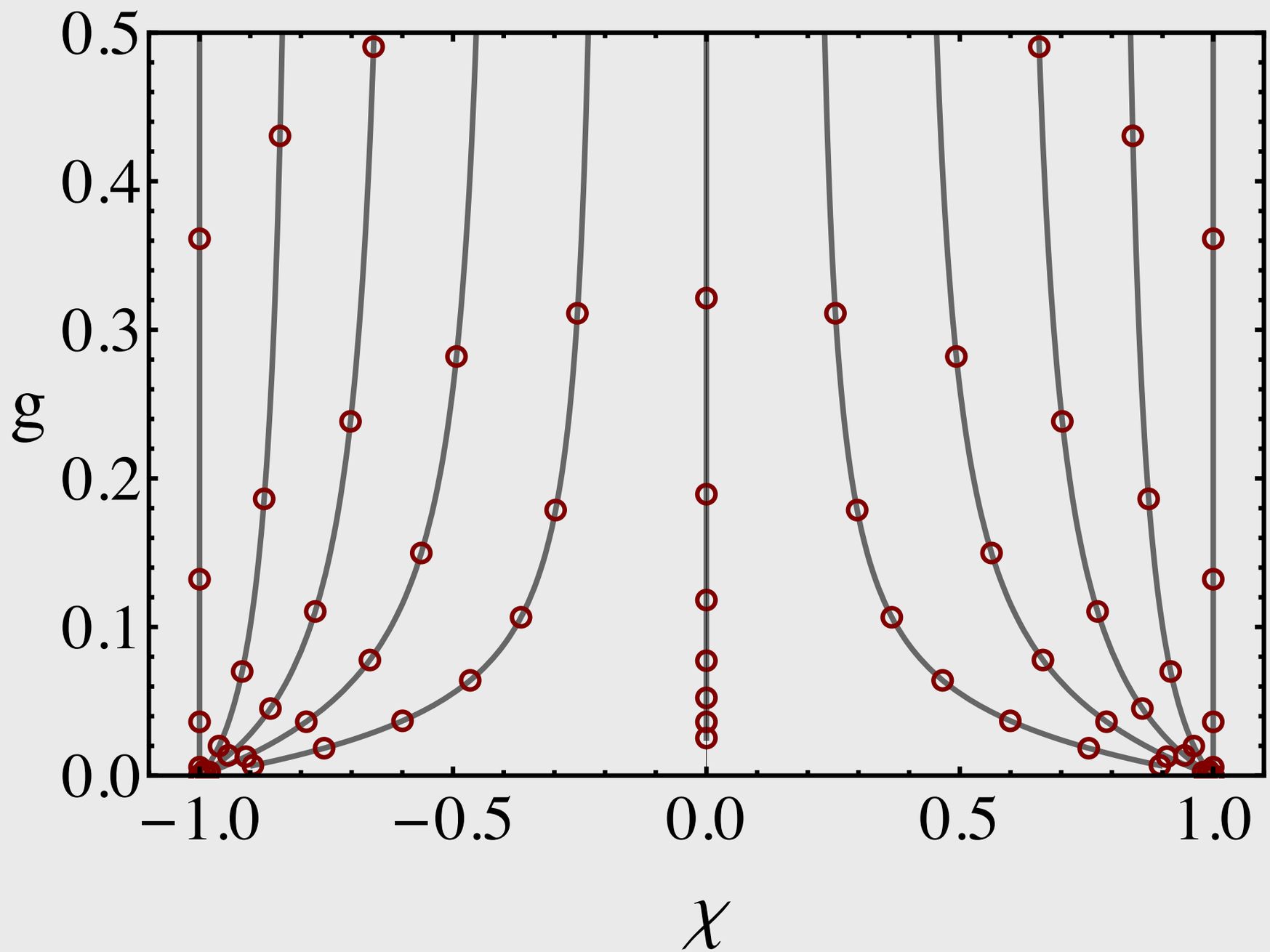
localization at large scales. In contrast, for class D anomalous "metallic" behavior already occurs in quasi-1d systems! By solving the $CI|DIII$ nonlinear sigma model using its quantum Hamiltonian, one finds that the (thermal) conductance decays algebraically for wires of arbitrary length. The anoma-

Zirnbauer, Serban, Bouquet, 99

transfer matrix cont'd

$$g(L) = \int dl\, l \coth\left(\pi l/2\right) \cos(\tilde{\xi} l L/\tilde{\xi}) e^{-l^2 L/\tilde{\xi}},$$

$$\xi(L) = \int dl\, \coth\left(\pi l/2\right) \sin(\tilde{\xi} l L/\tilde{\xi}) e^{-l^2 L/\tilde{\xi}},$$



1 and 2 dimensional topological insulators

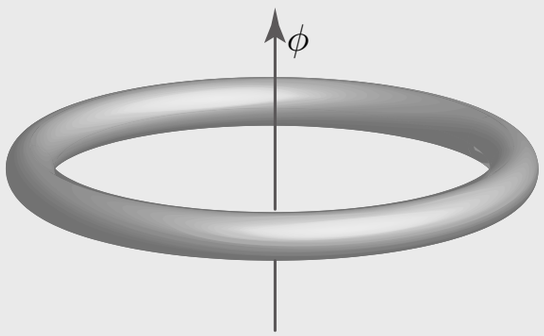
	A	AIII	AI	BDI	D	DIII	AII	CII	C	CI
1										
2										

$\mathbb{Z}$

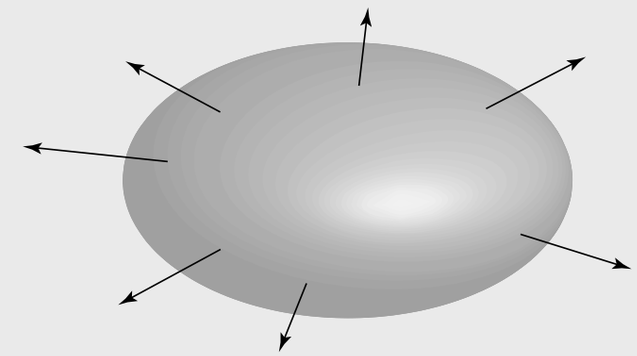
$\mathbb{Z}_2$

analogies (Z-insulators)

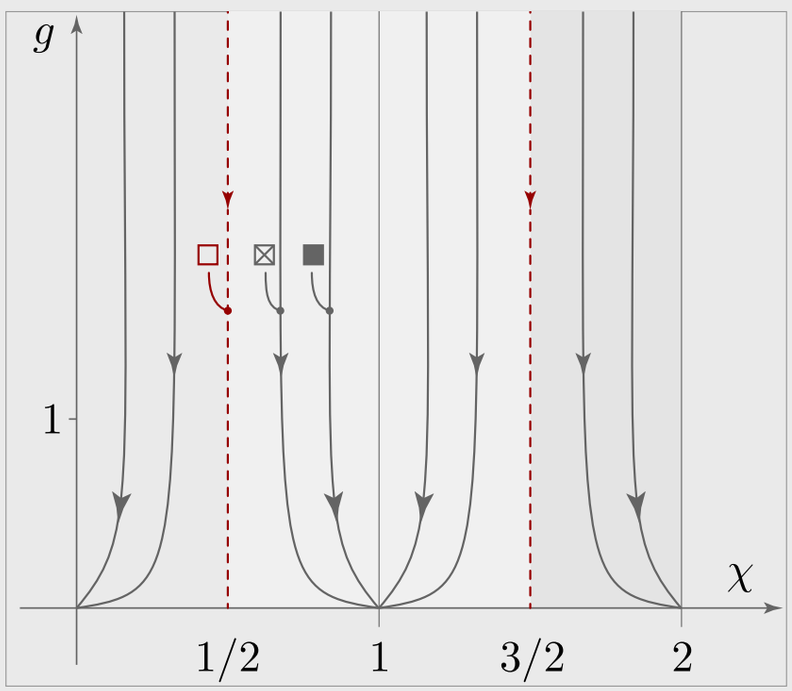
1d AIII, BDI, CII

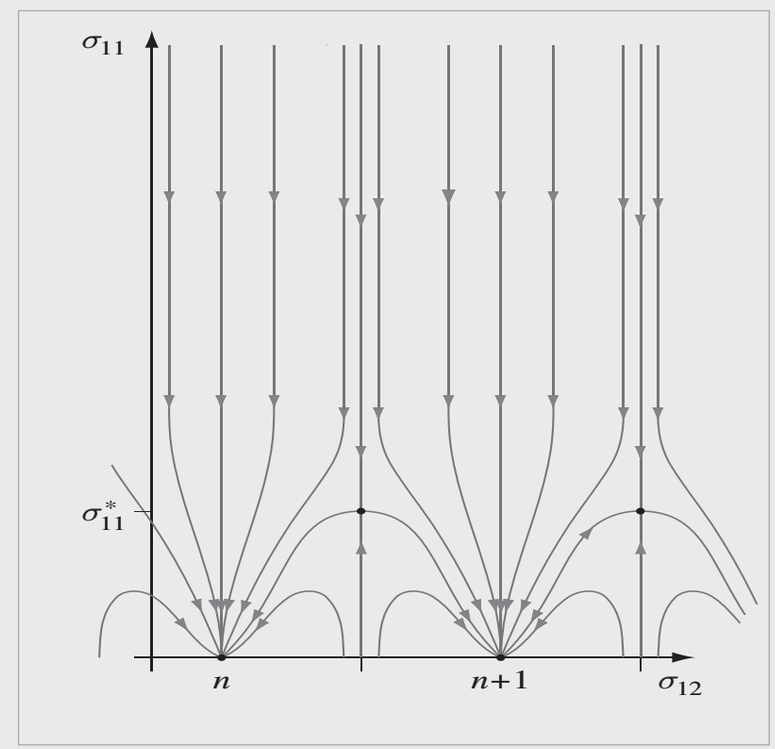


2d A, C, (D)



$$S[M] = \tilde{g}\, S_{\rm diff}[M] + \tilde{\chi}\, S_{\rm top}[M]$$





generic flow $(g, \chi) \overset{L\to\infty}{\longrightarrow} (0, n)$

$$n S_{\rm top}[M] \longrightarrow n\, S_{\rm boundary}[T]$$

field theory applications

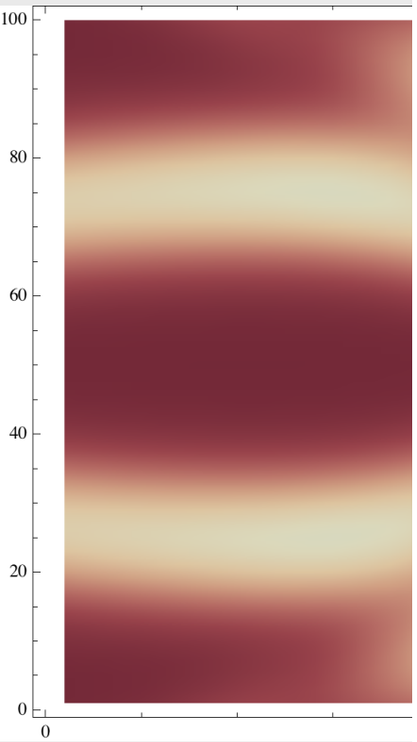
edge states

quantum transport

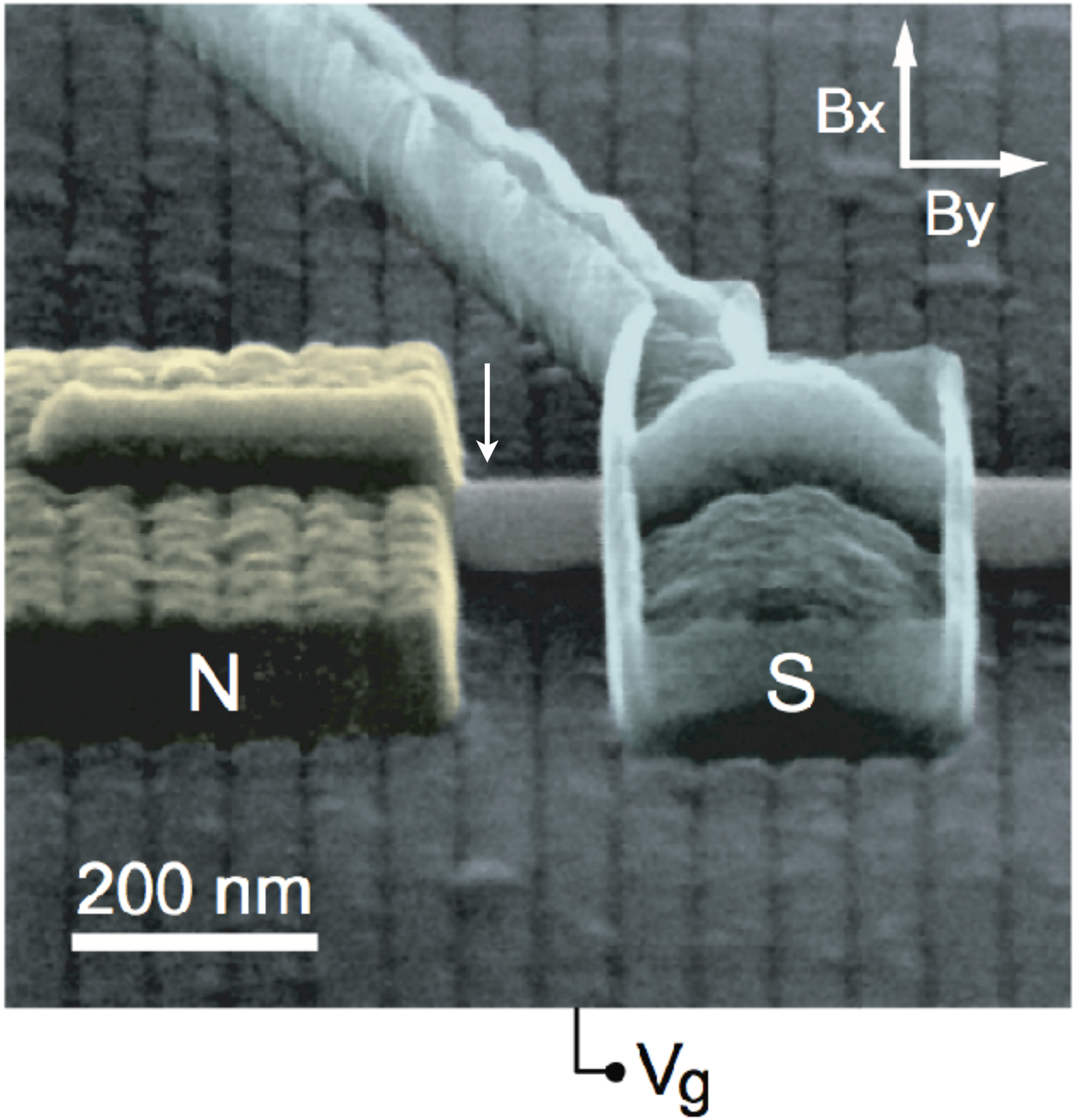
gapless topological matter

physics at the edge

experiment



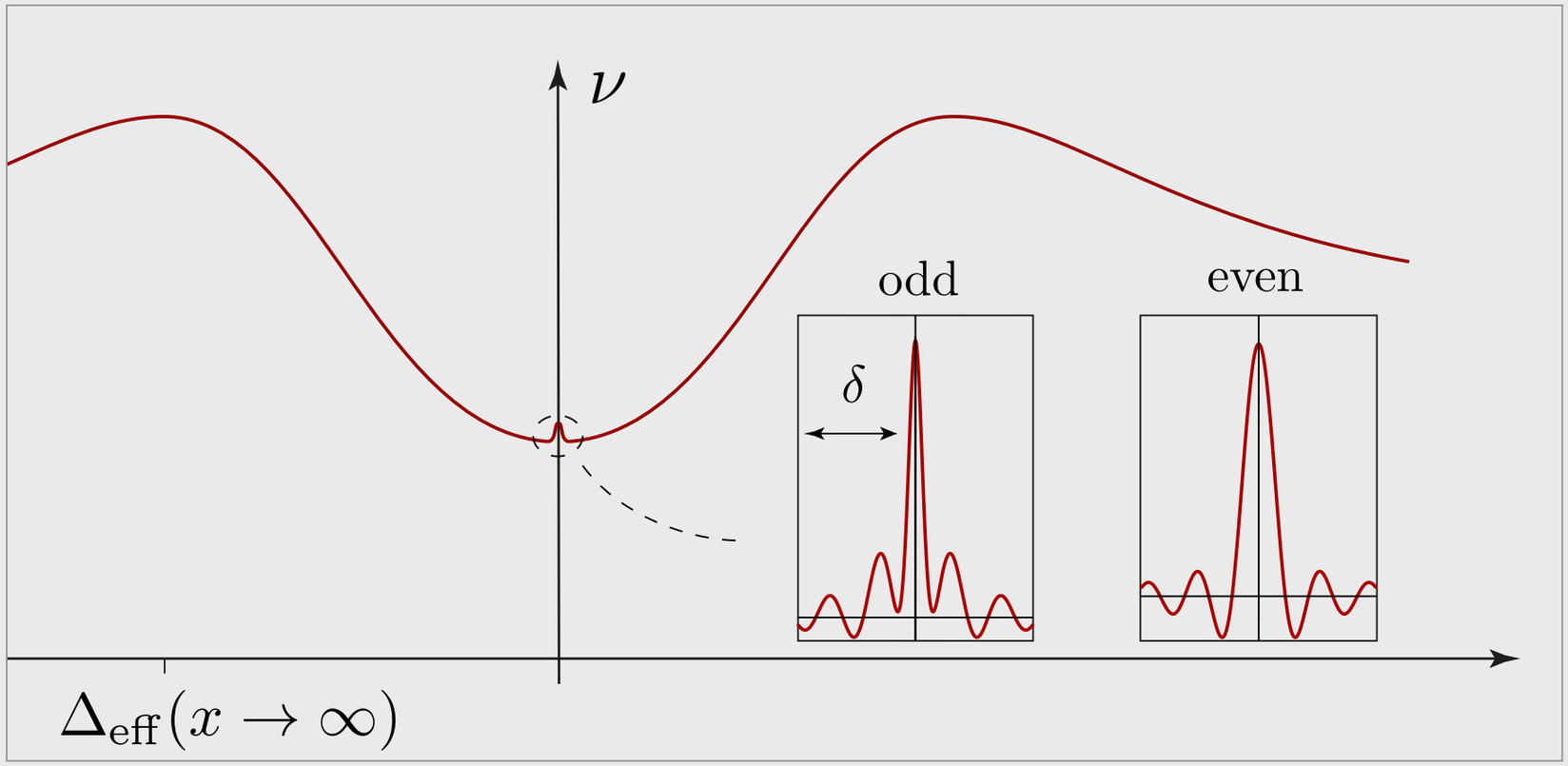
class D RMT



Churchill et al 2013

cf. A.A. & Bagrets, 12; Beenakker, 12; Lee & Parker, 12

field theory results for boundary DoS



$$N \text{ even}: \quad \frac{\nu(s)}{\nu_0} = 1 + \frac{\sin(s)}{s}, \qquad s = 2\pi \frac{\epsilon}{\delta}$$

level spacing

$$N \text{ odd}: \quad \frac{\nu(s)}{\nu_0} = 1 - \frac{\sin(s)}{s} + \delta\left(\frac{s}{2\pi}\right).$$

▷ topologically empty configuration exhibits spectral density similar to that of the topological state.

▷ universality of the DoS profile testifies to relevance beyond confines of the present model

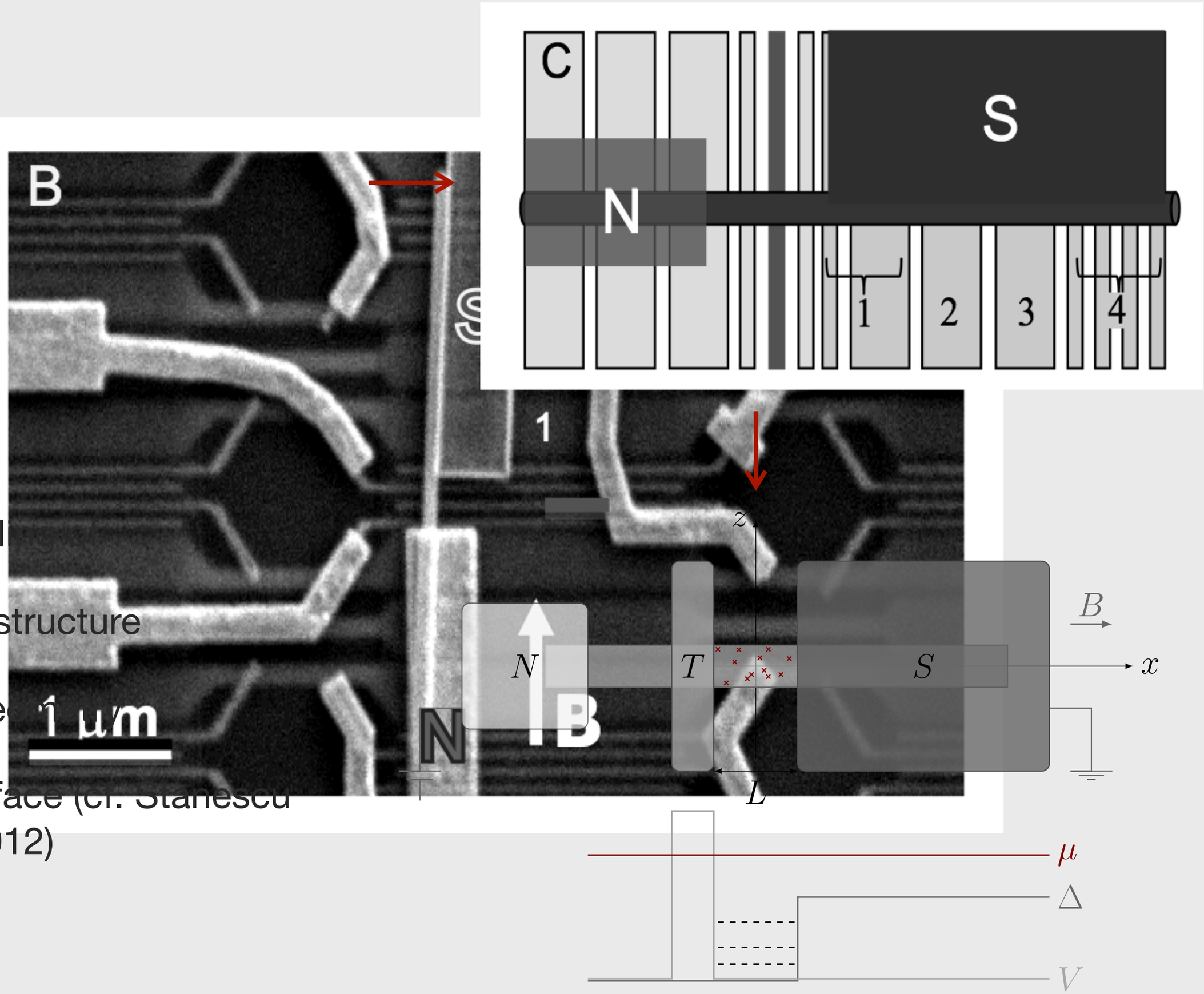
… reality

realistic model

+ channels structure

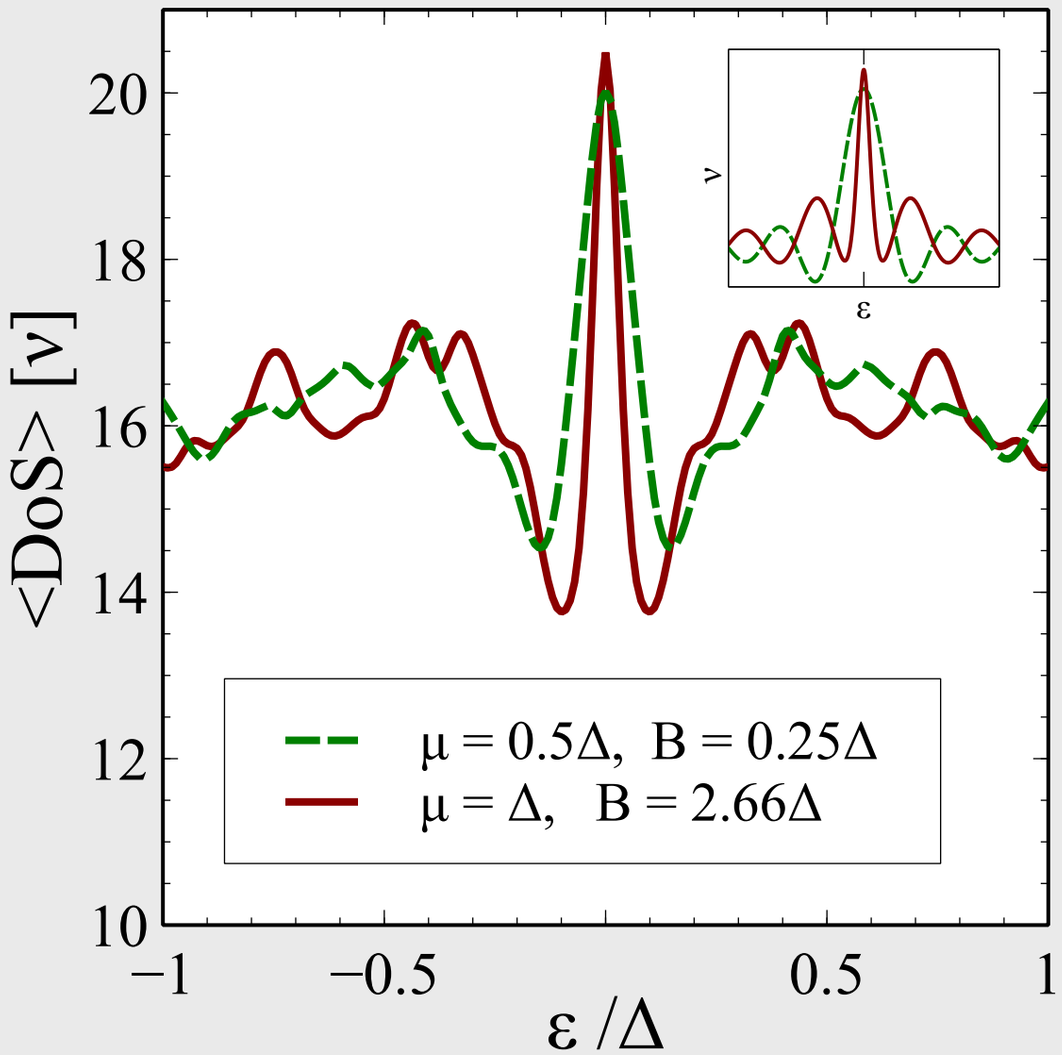
+ system ge

+ hard interface (cf. Stanescu & Tewari, 2012)

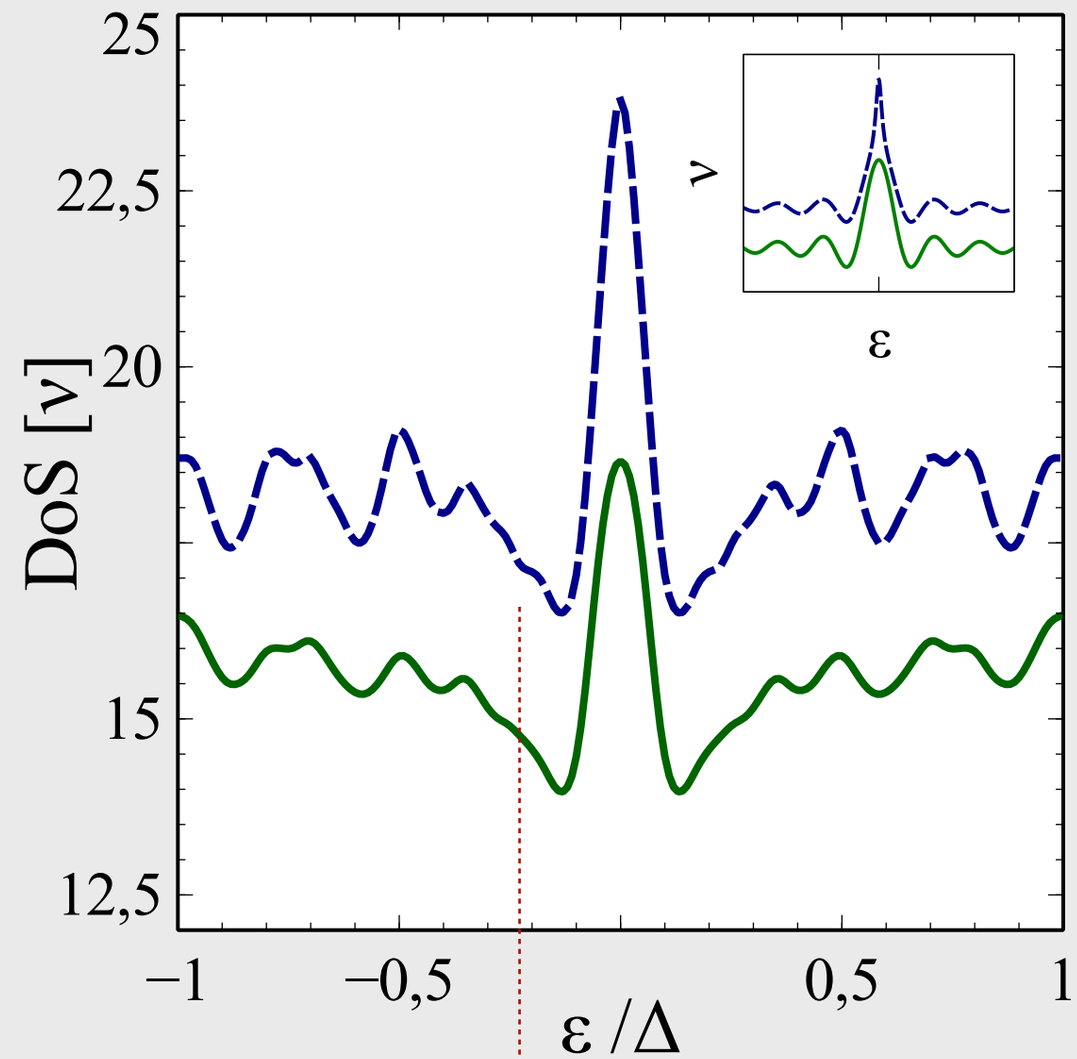


results

RMT comparison



average DoS



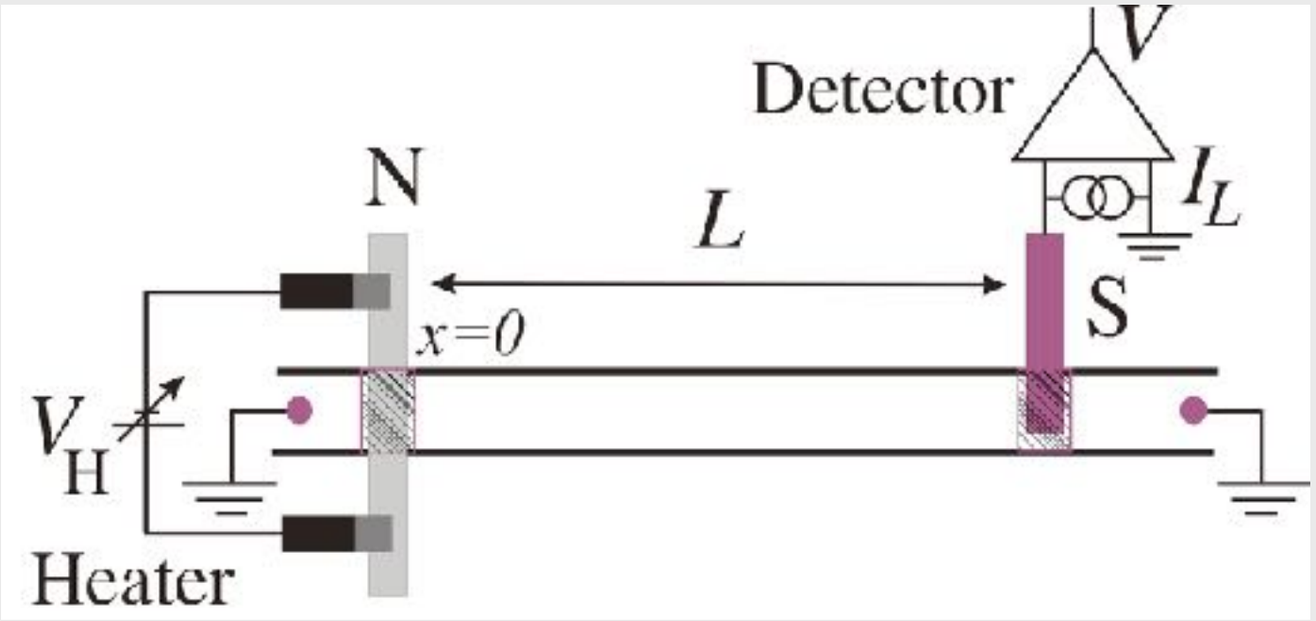
'typical DoS'

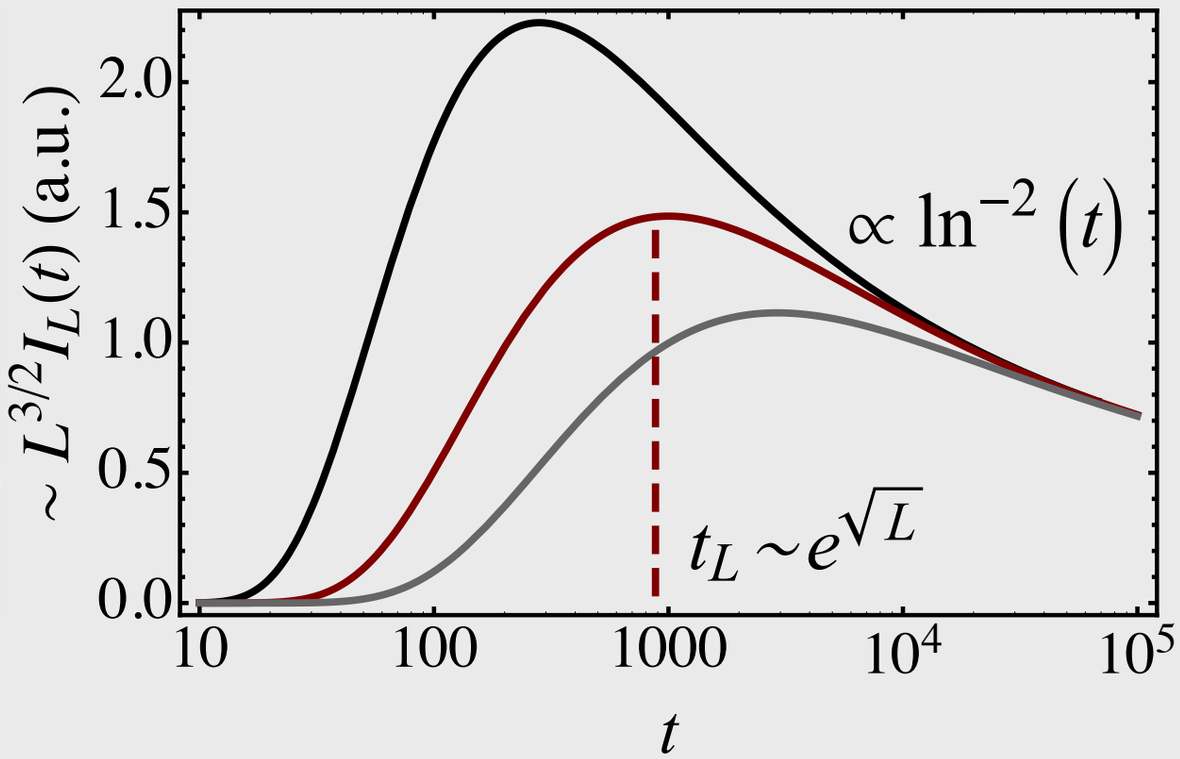
quantum critical transport

Q: @critical point between top phases: bulk gapless excitations. What is the dynamical critical exponent, *z*?

A: it is infinitely large. Rather than $x \propto Dt^{1/2}$ excitations propagate as

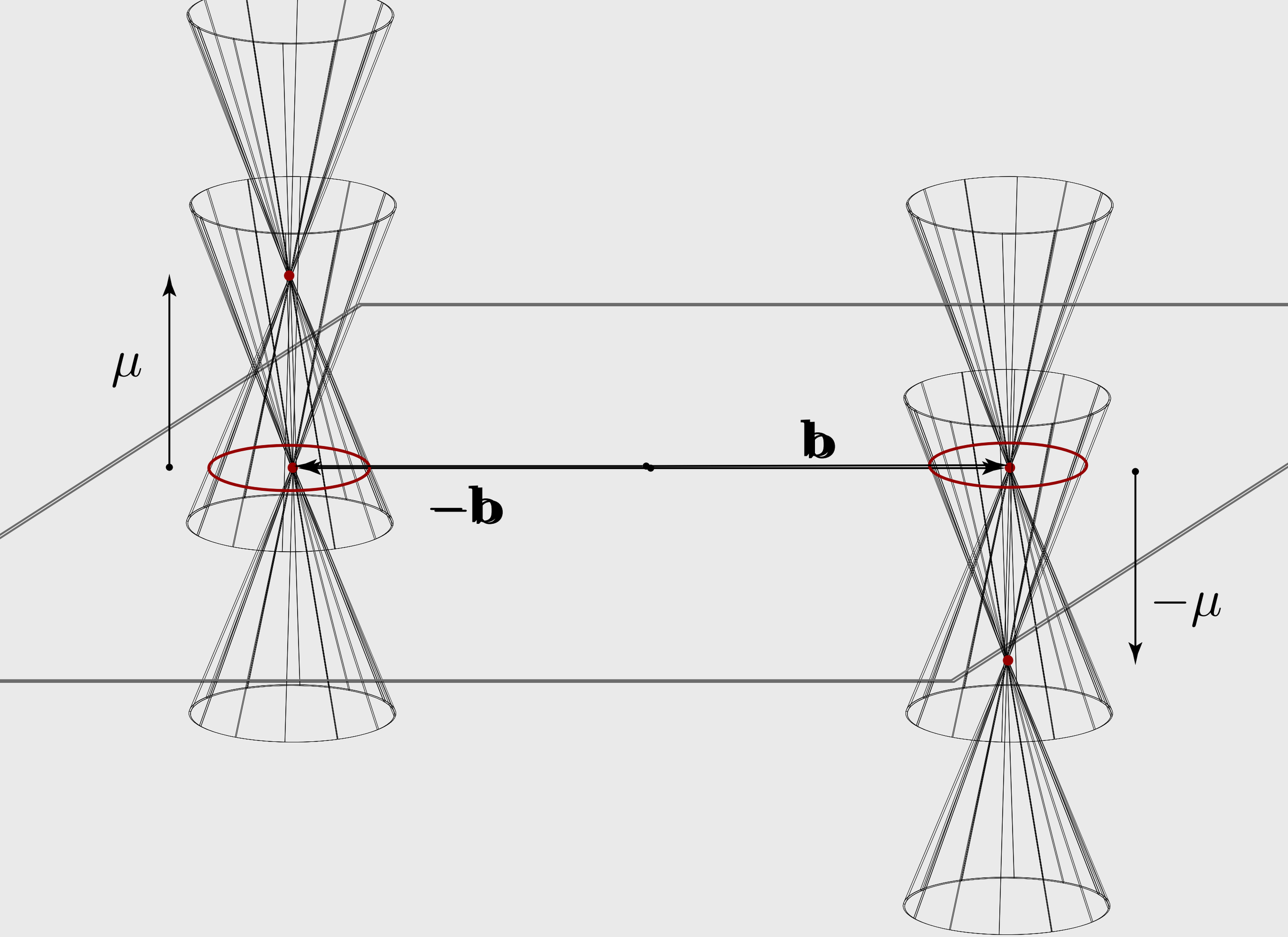
$$x \propto \xi_0 \log^2(t)$$

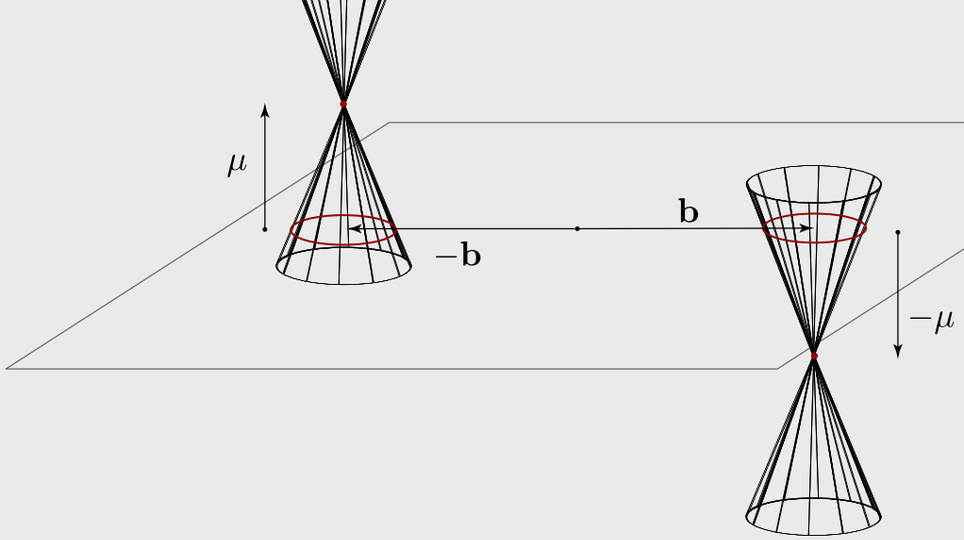




D. Bagrets, A.A, A. Kamenev, PRL 117, 196801 (2016)

disorder in gapless topological matter


μ
$\mathbf{b}$
$-\mathbf{b}$
$-\mu$



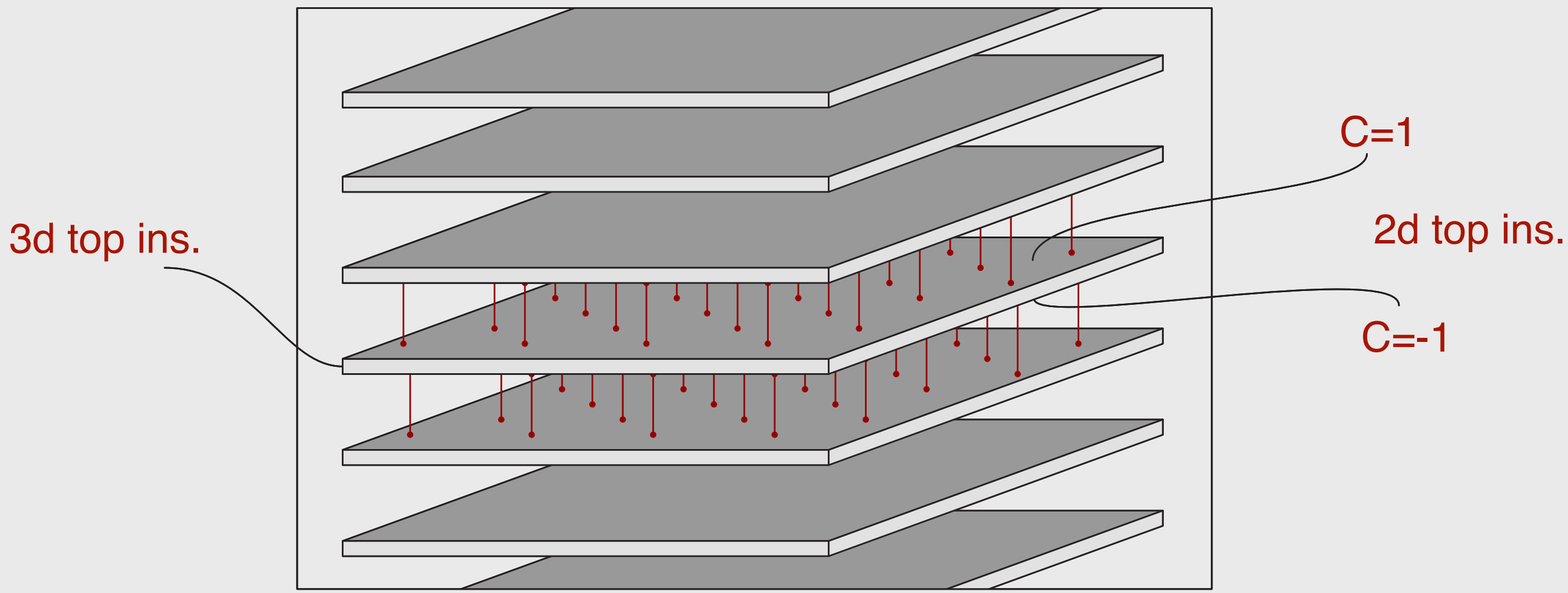
nodes protected against gapping

can be moved in energy/momentum

centers of topological Fermi surfaces

physical realization: stacked 2d topological insulators

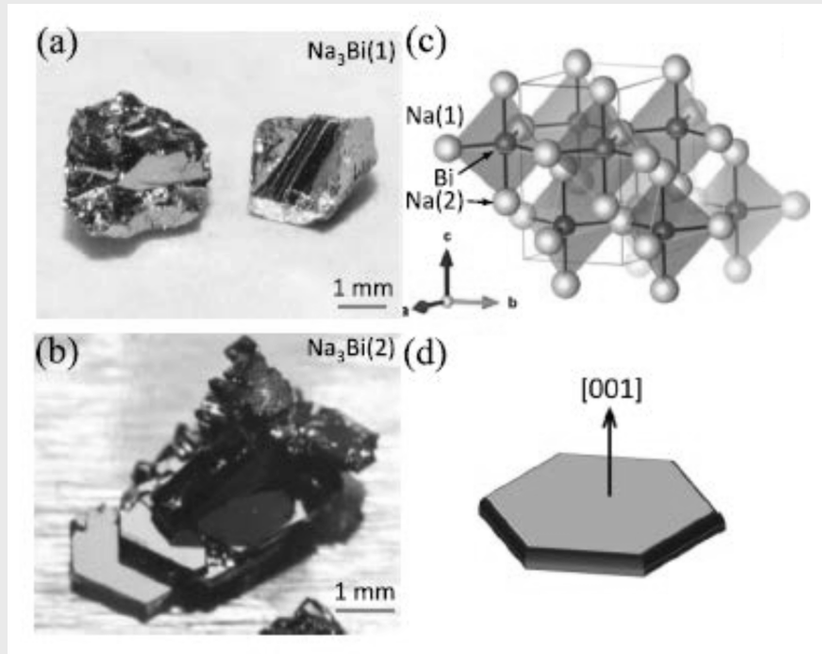
conceptual realization



Burkov and Balents, 11

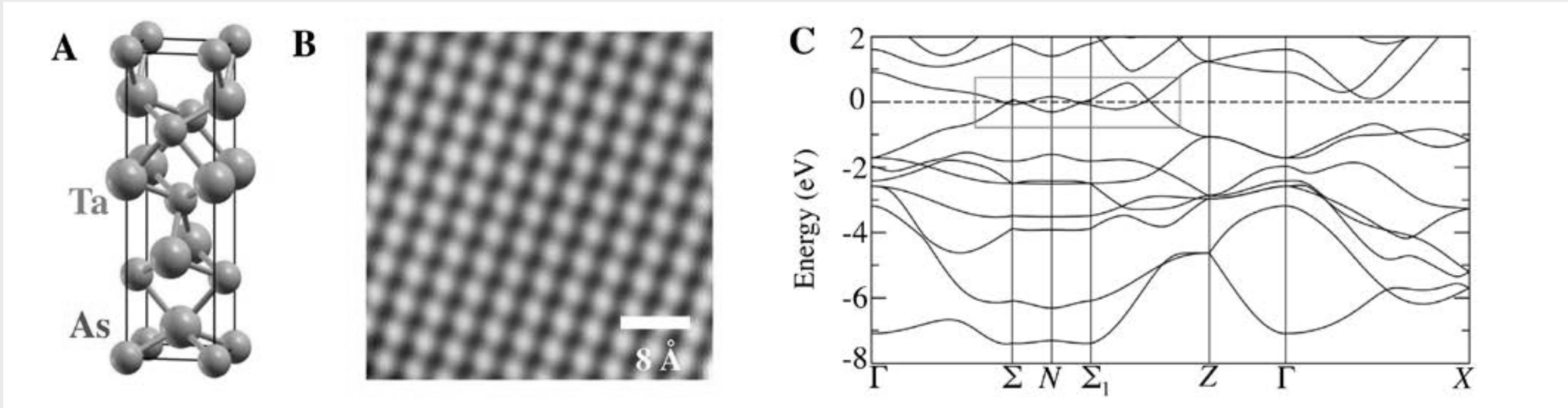
actual realization

Na_3Bi, Cd_3As_2 (Wang *et al*. 2012, theor., Liu *et al.* 2014, exp.)

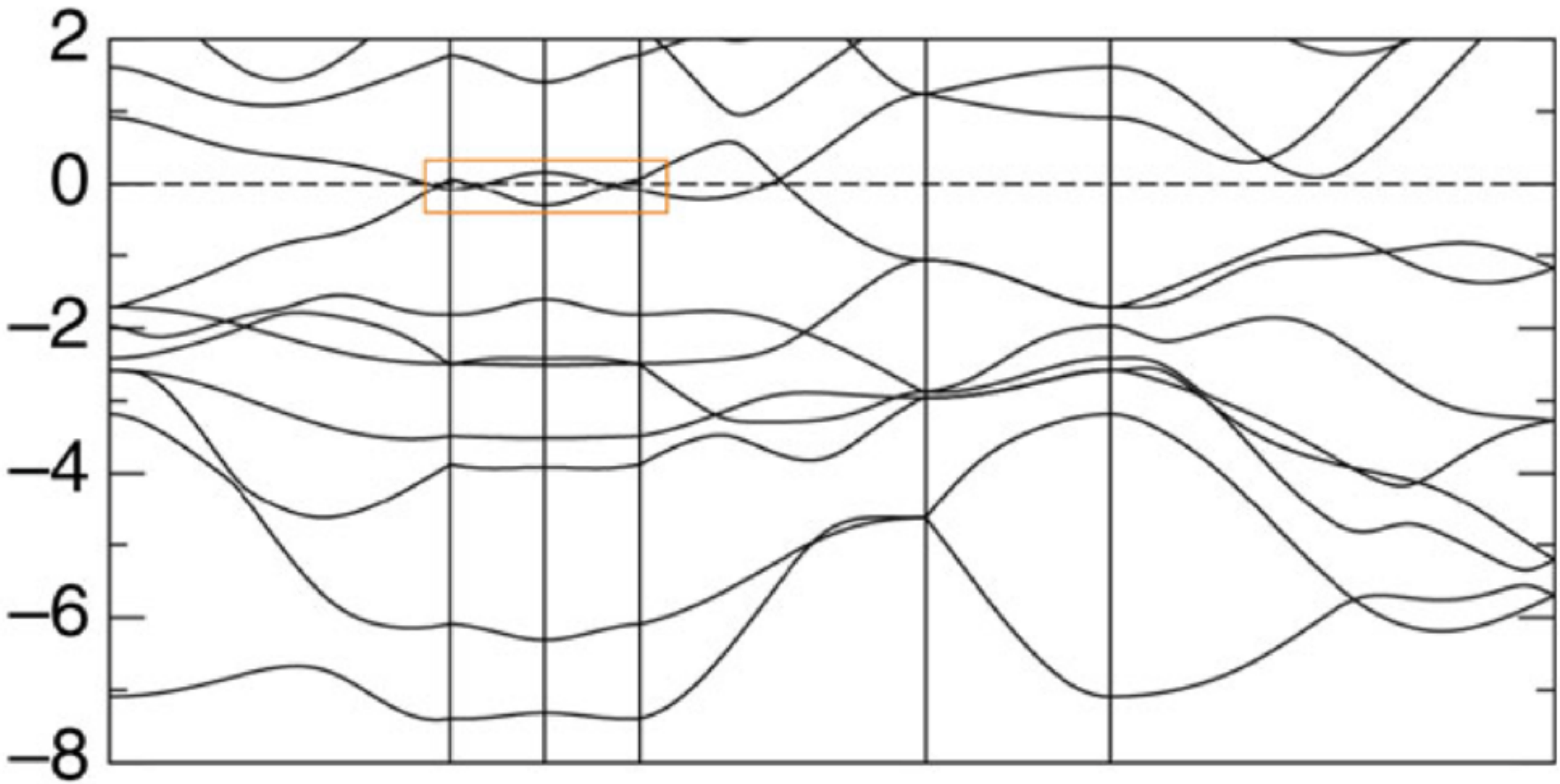


$HgCr_2Se_4$ (Xu *et al.* 2011, theor.)

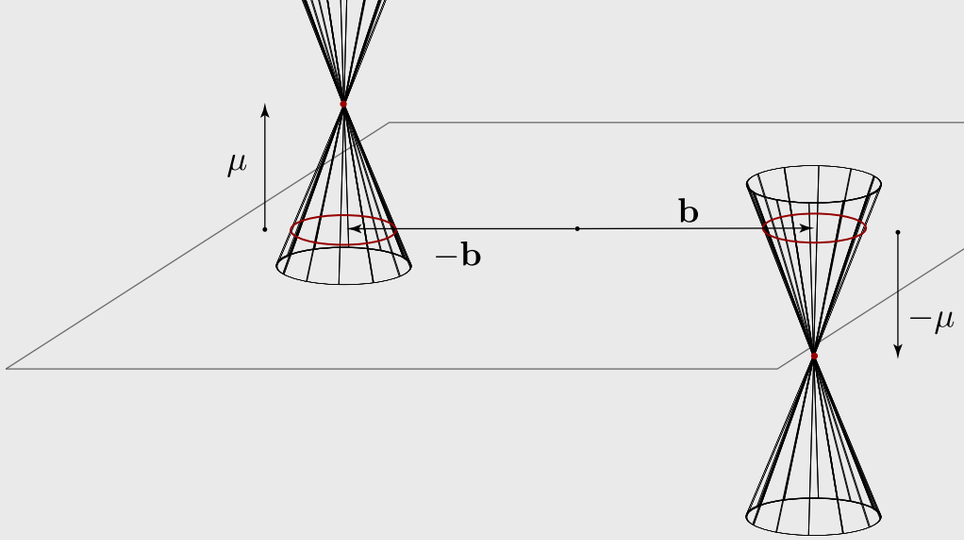
TaAs (Xu *et al.* 2011 theor., Weng *et al.* 2015 theor., Xu *et al.* 2015 exp.)



Band structure of TaAs



Huang *et al.* Nature comm. 2015



nodes protected against gapping

can be moved in energy/momentum

centers of topological Fermi surfaces

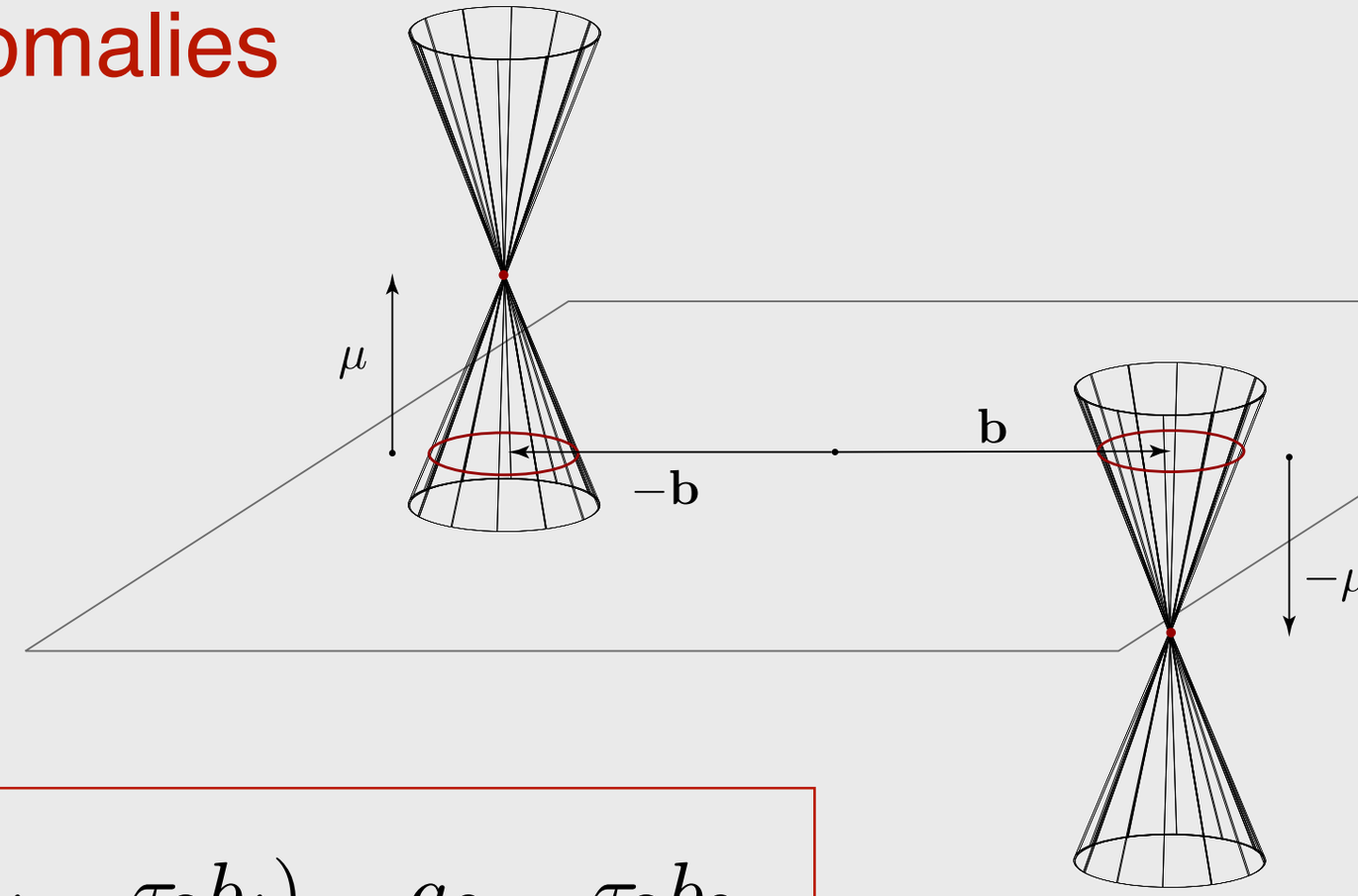
physical realization: stacked 2d topological insulators

surface states ('Fermi arcs')

non-conservation of charge at individual nodes (axial anomaly)

generates unconventional transverse response: chiral magnetic effect, CME and anomalous Hall effect, AHE

transverse response from anomalies



$$H = \tau_3 \otimes \sigma^i (k_i - a_i - \tau_3 b_i) - a_0 - \tau_3 b_0$$

τ_3 acts in nodal space

$a = (a_0, a_i)$ is external vector potential

$b = (b_0, b_i)$ is (3+1)-dimensional constant 'internal' vector potential. Coupled to axial current. Can be 'gauged out' by anomalous gauge transformation

transverse response from anomalies

axial current non-conservation

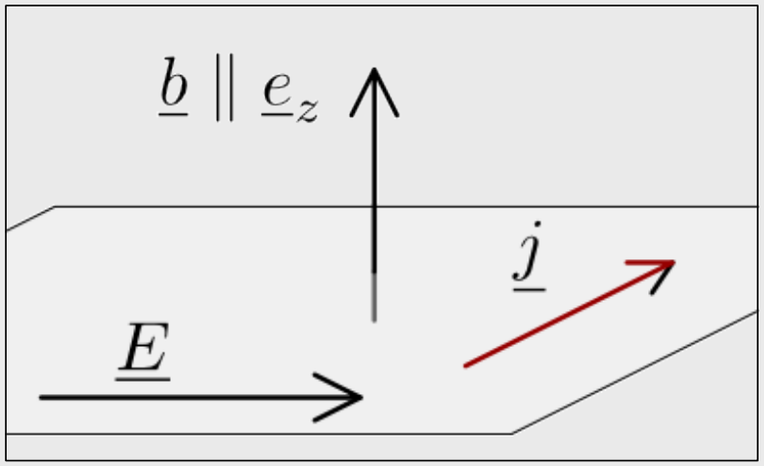
$$\partial_\mu j^{\mu,a} = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$j^a = j_{\text{node1}} - j_{\text{node2}}$$

leads to unconventional response coefficients

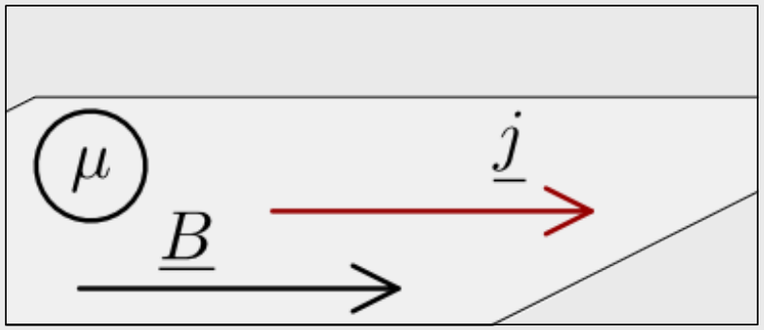
$$\underline{j} = \frac{1}{4\pi^2}\,\underline{b}\times\underline{E}$$

AHE



$$\underline{j} = \frac{1}{4\pi^2}\,\mu\underline{B}$$

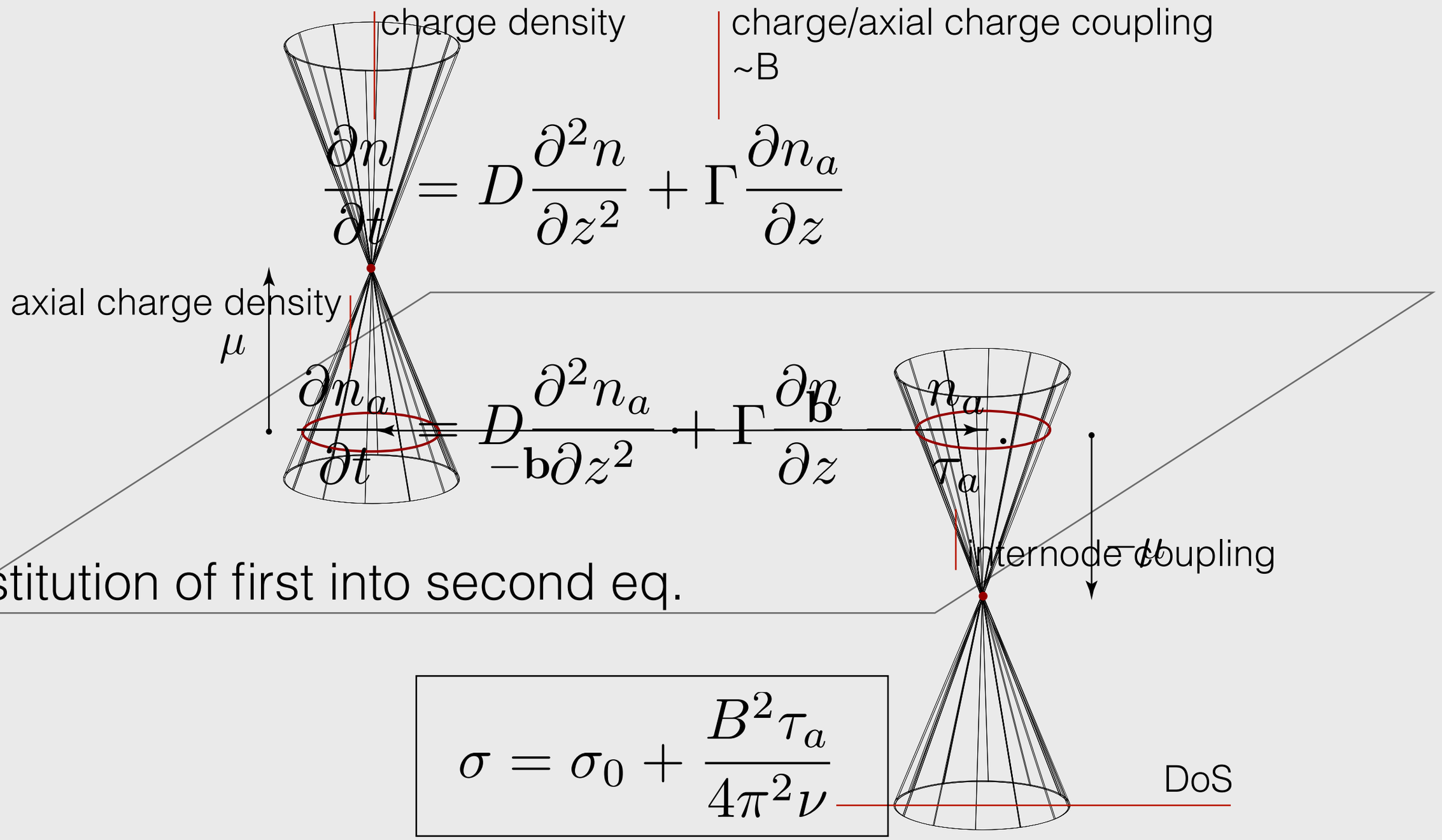
CME



disorder qualitative

diffusive transport

Q: do anomalies survive internode scattering? (Burkov et al. 14, Son & Spivak 13, $\mu \gg \tau^{-1}$)



$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial z^2} + \Gamma\frac{\partial n_a}{\partial z}$$

$$\frac{\partial n_a}{\partial t} = D\frac{\partial^2 n_a}{\partial z^2} + \Gamma\frac{\partial n}{\partial z} - \frac{n_a}{\tau_a}.$$

substitution of first into second eq.

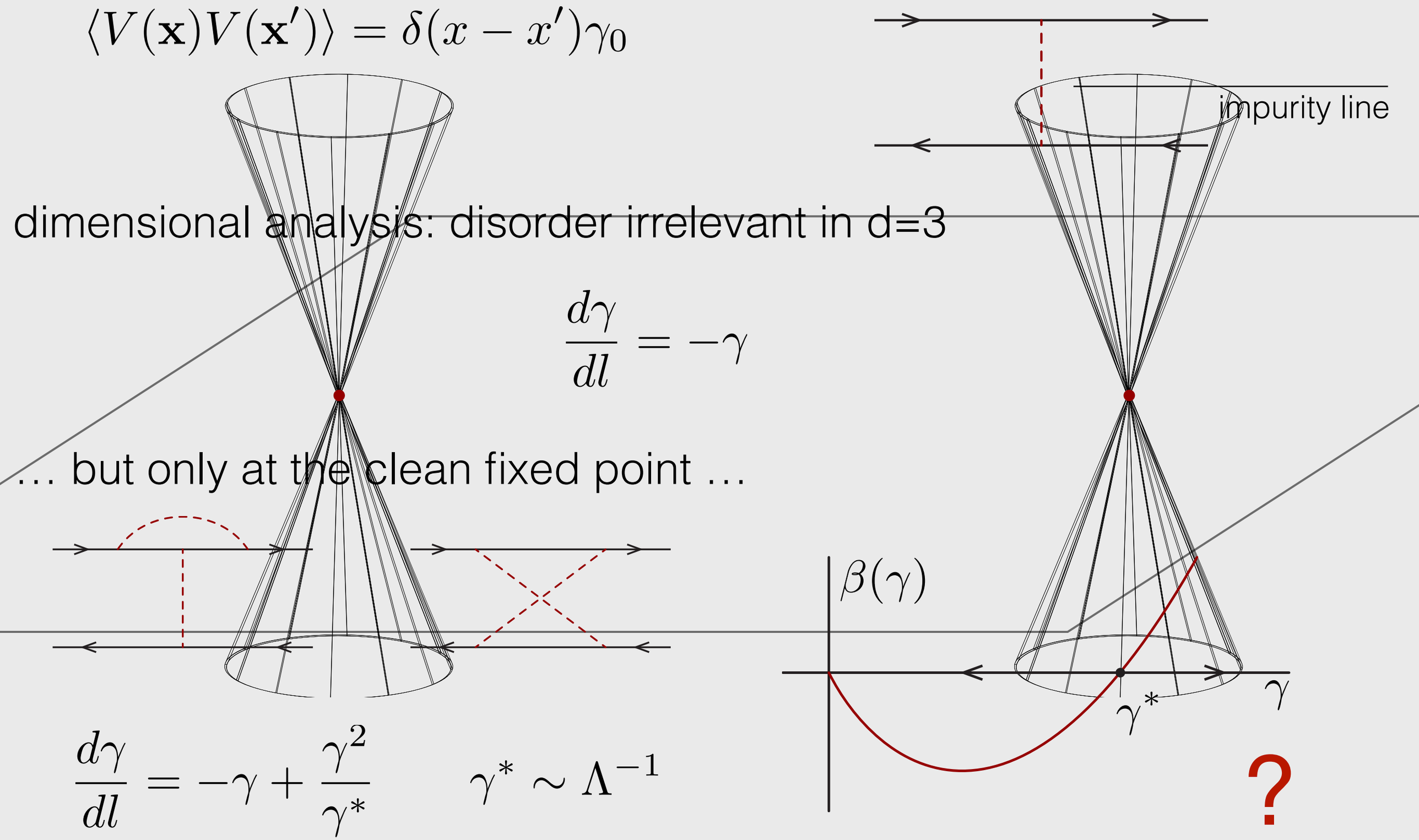
$$\sigma = \sigma_0 + \frac{B^2\tau_a}{4\pi^2\nu}$$

exp., Ong, 14: $\tau_a/\tau = \mathcal{O}(10^2)$

disorder at the Weyl nodes

Q: how does disorder affect the nodes? (Fradkin 86, Syzranov et al.14)

$$\langle V(\mathbf{x})V(\mathbf{x}')\rangle = \delta(x-x')\gamma_0$$



dimensional analysis: disorder irrelevant in d=3

$$\frac{d\gamma}{dl} = -\gamma$$

... but only at the clean fixed point ...

$$\frac{d\gamma}{dl} = -\gamma + \frac{\gamma^2}{\gamma^*} \qquad \gamma^* \sim \Lambda^{-1}$$

Q:

physics at large distance scales in the supercritically disordered system?

field theory

replica functional & disorder averaging

replica index
$a = 1, \ldots, R$

$$\langle G^+ G^- \ldots \rangle \longleftrightarrow Z = \int D(\bar{\psi}, \psi) \left\langle e^{i\bar{\psi}^a (\hat{\mu} + i\delta\tau_3 - \hat{H})\psi^a} \right\rangle$$

discriminates
between retarded and advanced

replica rotation symmetry: action invariant under

$$\boxed{\psi \to T\psi, \qquad T \in \mathrm{U}(2R)}$$

stationary phase analysis

disorder averaged functional

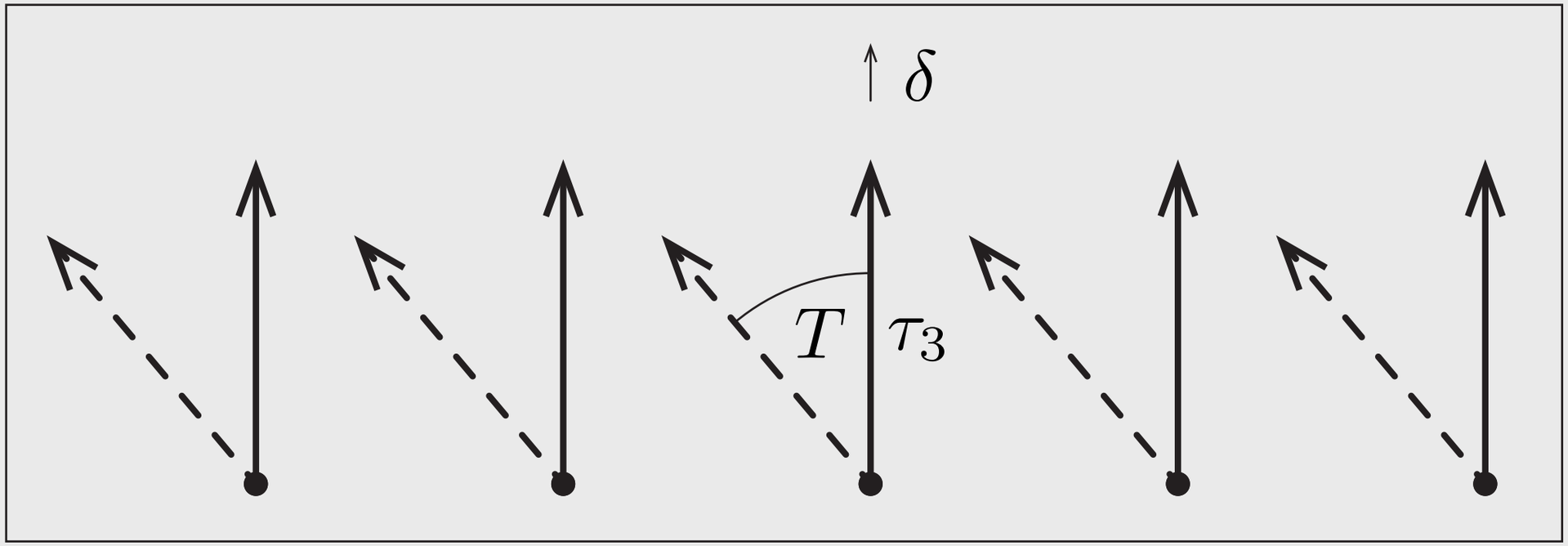
$$Z = \int D(\bar{\psi}, \psi) DT \, e^{i\bar{\psi}(\hat{\mu} + i\kappa T \tau_3 T^{-1} - \hat{\hat{H}})\psi}$$

$\kappa = (2/\pi) v \Lambda (1 - \gamma^*/\gamma)$

Fradkin 86

mean scattering rate

replica rotation symmetry spontaneously broken *if* system is supercritically disordered



anomaly and gauge structure

$$Z = \int D(\bar{\psi}, \psi) DT \, e^{i\bar{\psi}(\mu + i\kappa T \tau_3 T^{-1} - \hat{H})\psi}$$

expand action in fluctuations $A_i = T^{-1}\partial_i T$

Goldstone modes

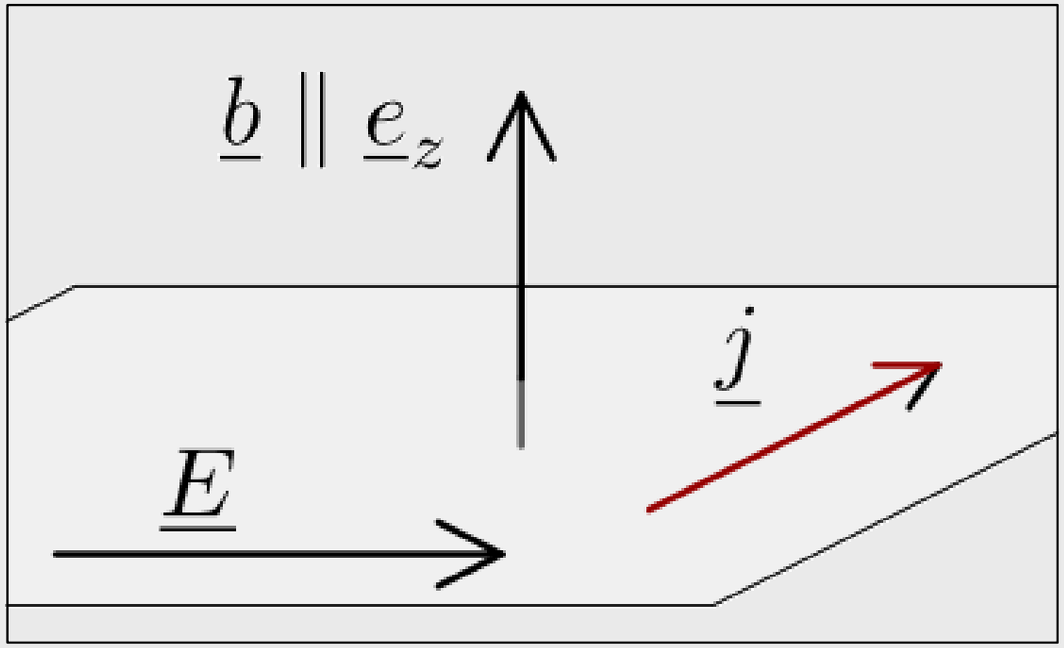
the main actors

gauge field — parity anomaly

Redlich 84

$$S_{\text{eff}}[A] = S^{(1)}[A] + S^{(2)}[A] + S^{(3)}[A] + \ldots$$

discussion I: AHE



layered quantum Hall action

$$S_{\mathrm{d}}[A] = \frac{\sigma_{xx}}{8} \sum_i \int d^3x \, \mathrm{tr}([A_i, \tau_3]^2),$$

$$S_{\mathrm{top}}[A] = -\frac{\sigma_{xy}}{2} \epsilon^{3ij} \int d^3x \, \mathrm{tr}(\tau_3 \partial_i A_j),$$

$$\Big\downarrow \; Q = T \tau_3 T^{-1}$$

$$\boxed{\begin{aligned} S_{\mathrm{d}}[Q] &= \frac{\sigma_{xx}}{8} \int d^3x \, \mathrm{tr}(\partial Q^2) \\ S_{\mathrm{top}}[Q] &= -\frac{\sigma_{xy}}{8} \epsilon^{3ij} \int d^3x \, \mathrm{tr}(Q \partial_i Q \partial_j Q) \end{aligned}}$$

layered quantum Hall action

$$S_{\mathrm{d}}[Q] = \frac{\sigma_{xx}}{8} \int d^3x \, \mathrm{tr}(\partial Q^2)$$

$$S_{\mathrm{top}}[Q] = -\frac{\sigma_{xy}}{8} \epsilon^{3ij} \int d^3x \, \mathrm{tr}(Q \partial_i Q \partial_j Q)$$

action describes layered quantum Hall systems

Three-Dimensional Disordered Conductors in a Strong Magnetic Field: Surface States and Quantum Hall Plateaus

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(Received 6 July 1995)

We study localization in layered, three-dimensional conductors in strong magnetic fields. We demonstrate the existence of three phases—insulator, metal, and quantized Hall conductor—in the two-dimensional parameter space obtained by varying the Fermi energy and the interlayer coupling strength. Transport in the quantized Hall conductor occurs via extended surface states. These surface states constitute a subsystem at a novel critical point, which we describe using a new, directed network model.

layered quantum Hall action

$$S_{\mathrm{d}}[Q] = \frac{\sigma_{xx}}{8}\int d^3x\,\mathrm{tr}(\partial Q^2)$$

$$S_{\mathrm{top}}[Q] = -\frac{\sigma_{xy}}{8}\epsilon^{3ij}\int d^3x\,\mathrm{tr}(Q\partial_i Q\partial_j Q)$$

coupling constants

$$\sigma_{xx} = \frac{\mu^2+3\kappa^2}{6\pi\kappa v} \longrightarrow \begin{cases} \frac{\kappa}{2\pi v} & \mu \to 0 \\ \frac{1}{3}v^2\gamma & \mu \gg \kappa \end{cases}$$

longitudinal conductivity

$$\sigma_{xy} = \frac{1}{\mathrm{Vol}}\partial_B N \longrightarrow \frac{1}{L_z}\sum_n C_n \xrightarrow{L_z b \gg 1} \frac{b}{2\pi}$$

Hall conductivity

disorder averaged Chern number of n-th layer

layered quantum Hall action

$$S_{\mathrm{d}}[Q] = \frac{\sigma_{xx}}{8} \int d^3x \, \mathrm{tr}(\partial Q^2)$$

$$S_{\mathrm{top}}[Q] = -\frac{\sigma_{xy}}{8} \epsilon^{3ij} \int d^3x \, \mathrm{tr}(Q \partial_i Q \partial_j Q)$$

Localization and Metal-Insulator Transition in Multilayer Quantum Hall Structures

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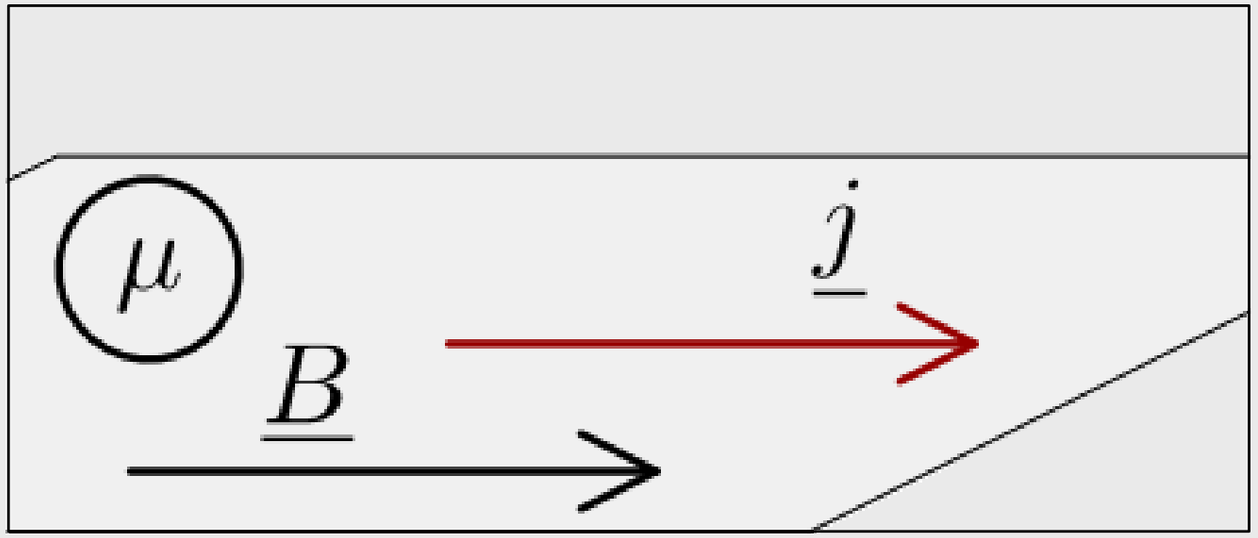
(Received 10 June 1997)

We study the phase structure and Hall conductance quantization in weakly coupled multilayer electron systems in the integer quantum Hall regime. We derive an effective field theory and perform a two-loop renormalization group calculation. It is shown that (i) finite interlayer tunnelings (however small) give rise to successive metallic and insulating phases and metal-insulator transitions in the unitary universality class; (ii) the Hall conductivity is not renormalized in the metallic phases in the 3D regime; and (iii) the Hall conductances are quantized in the insulating phases. In the bulk quantum Hall phases, the effective field theory describes the transport on the surface. [S0031-9007(97)04575-4]

discussion II: CME



CS action

higher order expansion in A/derivatives

$$S_{\mathrm{CS}}[A] = -\frac{i\epsilon^{ijk}}{8\pi}\sum_{s=\pm} s\int d^3x\,\mathrm{tr}\left(A_i P^s \partial_j A_k P^s + \frac{2}{3}A_i P^s A_j P^s A_k P^s\right)$$

projector onto
retarded/advanced sector

building faith in CS action

quadratic expansion reproduces axial diffusion propagator

$$T = \exp(W)$$

$$A_i = T^{-1}\partial_i T = \partial_i W - \frac{1}{2}[W, \partial_i W] + \dots$$

$$W = \begin{pmatrix} & C \\ -C^\dagger & \end{pmatrix}$$

quadratic action in external magnetic field *B*

quadratic expansion con't

from CS action

DoS

Pauli in nodal space

$$S^{(2)}[C, C^\dagger] = \frac{\nu}{2\pi} \int d^3x \, \mathrm{tr} \left(C \left(D\partial^2 + i\omega + \frac{1}{\tau_a}(1 + \tau_1^n) - i\frac{B}{\nu}\tau_3^n \right) C^\dagger \right)$$

diffusion constant

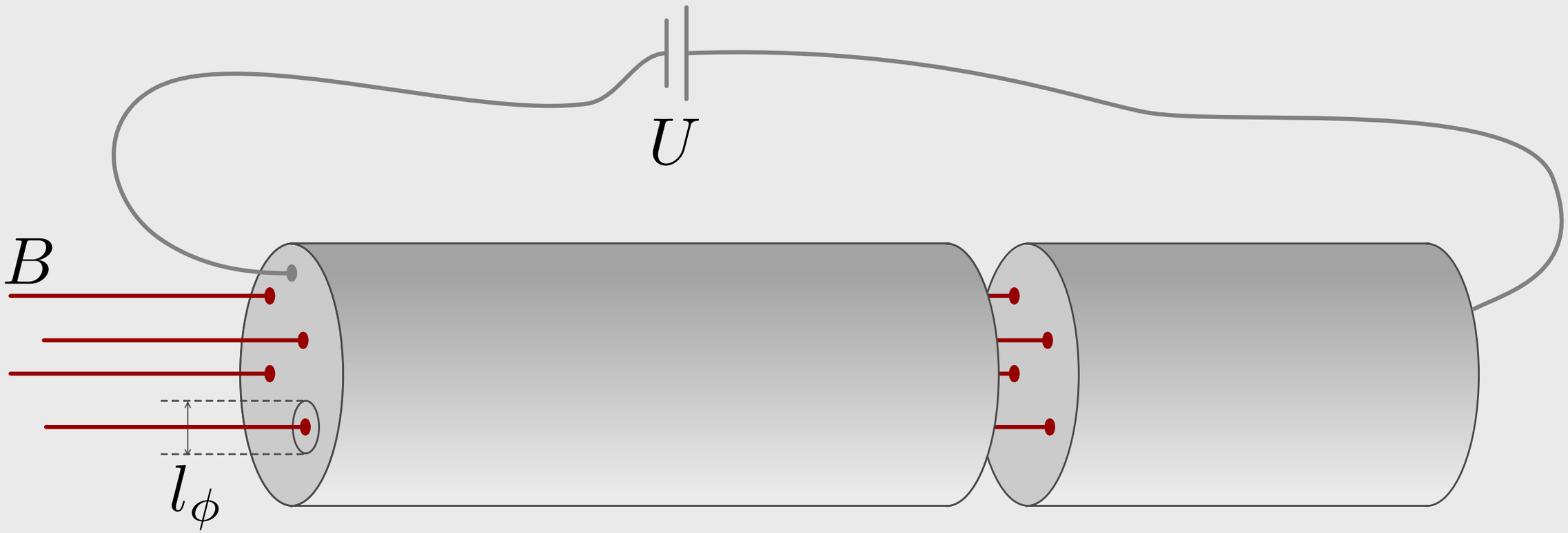
internode scattering time

produces axial diffusion mode

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial z^2} + \Gamma\frac{\partial n_a}{\partial z} \qquad \Gamma = \frac{B}{4\pi^2\nu}$$

$$\frac{\partial n_a}{\partial t} = D\frac{\partial^2 n_a}{\partial z^2} + \Gamma\frac{\partial n}{\partial z} - \frac{n_a}{\tau_a}.$$

quantum transport



Magnetoconductance?

Burkov *et al.*, Spivak & Son: $g = g_0 + \dfrac{B^2 \tau_a}{4\pi^2 L}$

Parameswaran *et al.*: $g = B L_\perp^2 = N_\phi$

CS action projected to 1d

enter with configuration

$$A_i = T^{-1} a_i T + T^{-1} \partial_i T$$

external field/sources

into CS action and assume $L_z \gg L_{x,y}$

$$S[T] = \sum_{n=1,2} \int dz \Big(\frac{\pi D S \nu}{4} \mathrm{tr}([T_n^{-1} \partial_z T_n, \sigma_3]^2) +$$

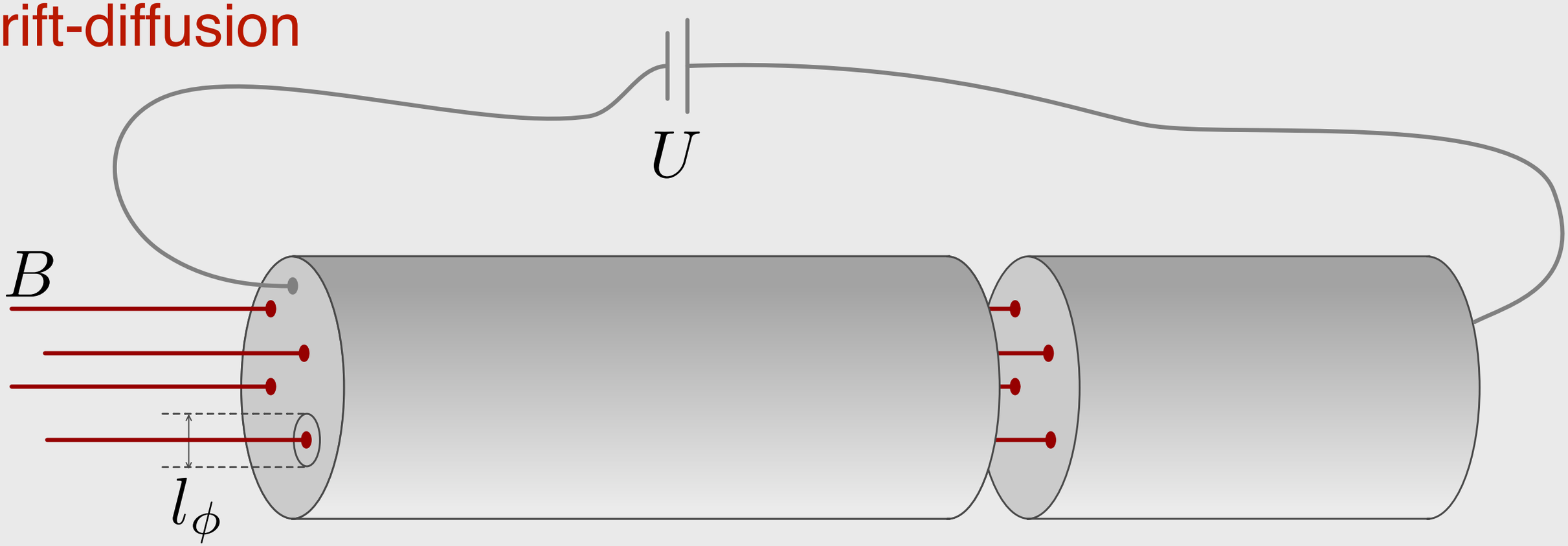
nodes

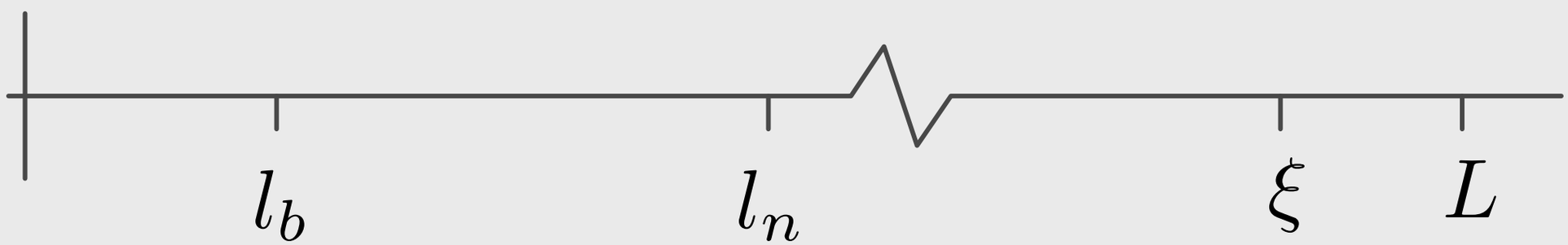
$$(-)^n \frac{N_\phi}{\pi} \mathrm{tr}(T_n^{-1} \partial_z T_n \sigma_3) \Big) +$$

wire cross section

$$\frac{S}{\nu \tau_a} \int dz \, \mathrm{tr}(T_1 \sigma_3 T_1^{-1} \, T_2 \sigma_3 T_2^{-1})$$

drift-diffusion





drift/diff. crossover

$$l_b = \frac{D\nu}{B}$$

nodal hybridization

$$l_n = (D\tau_a)^{1/2}$$

localization

conductance

compute conductance in diffusion approximation $(L \ll \xi)$

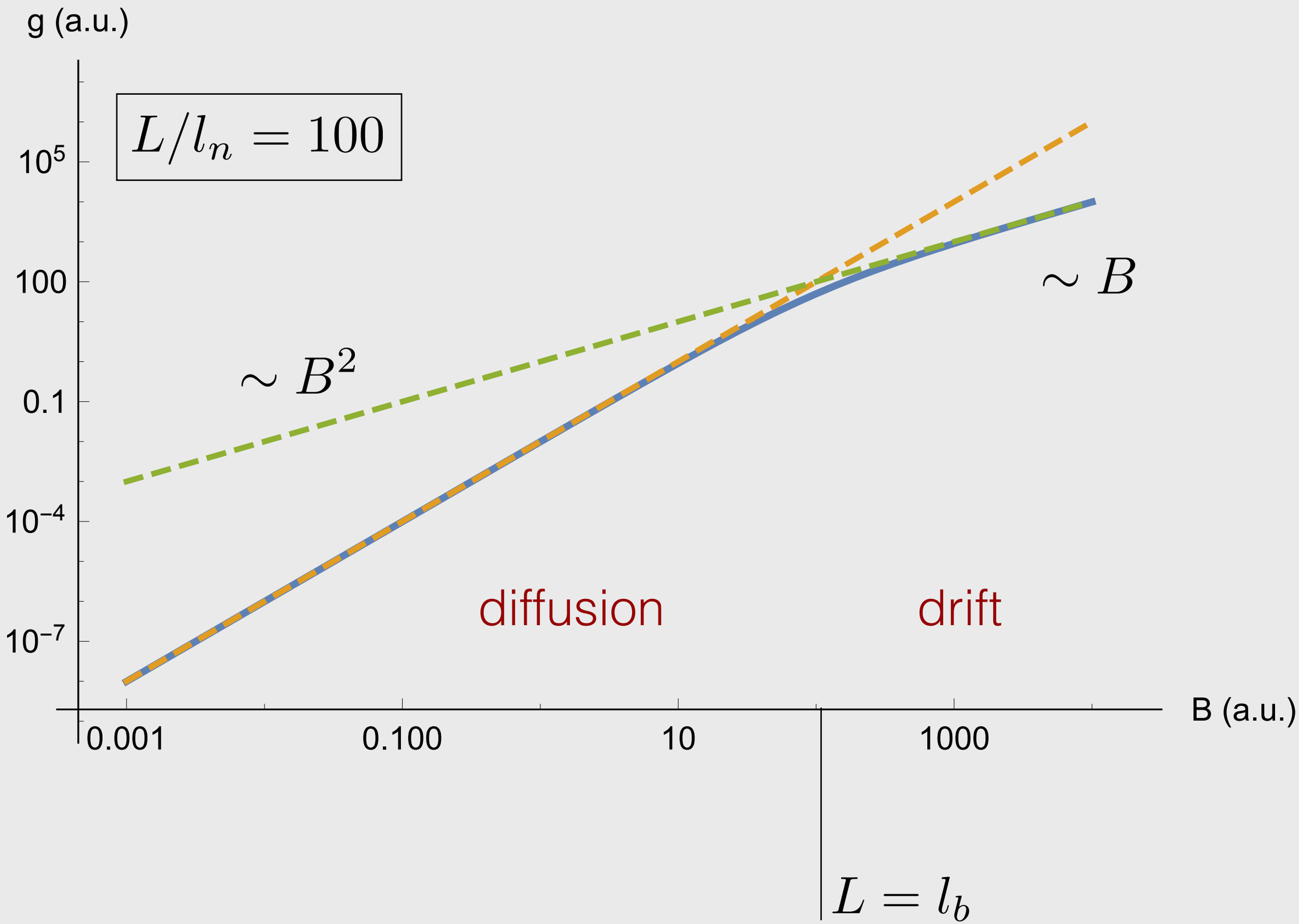
$$g = \frac{N_\phi}{2\pi} \frac{r(1+r^{-2})^{\frac{3}{2}}}{\tilde{L}(1+r^{-2})^{\frac{1}{2}} + r^{-2}\tanh\left(\frac{\tilde{L}}{2}(1+r^{-2})^{\frac{1}{2}}\right)}$$

$$\tilde{L} \equiv \frac{L}{l_n} \qquad r \equiv \frac{l_b}{l_n}$$

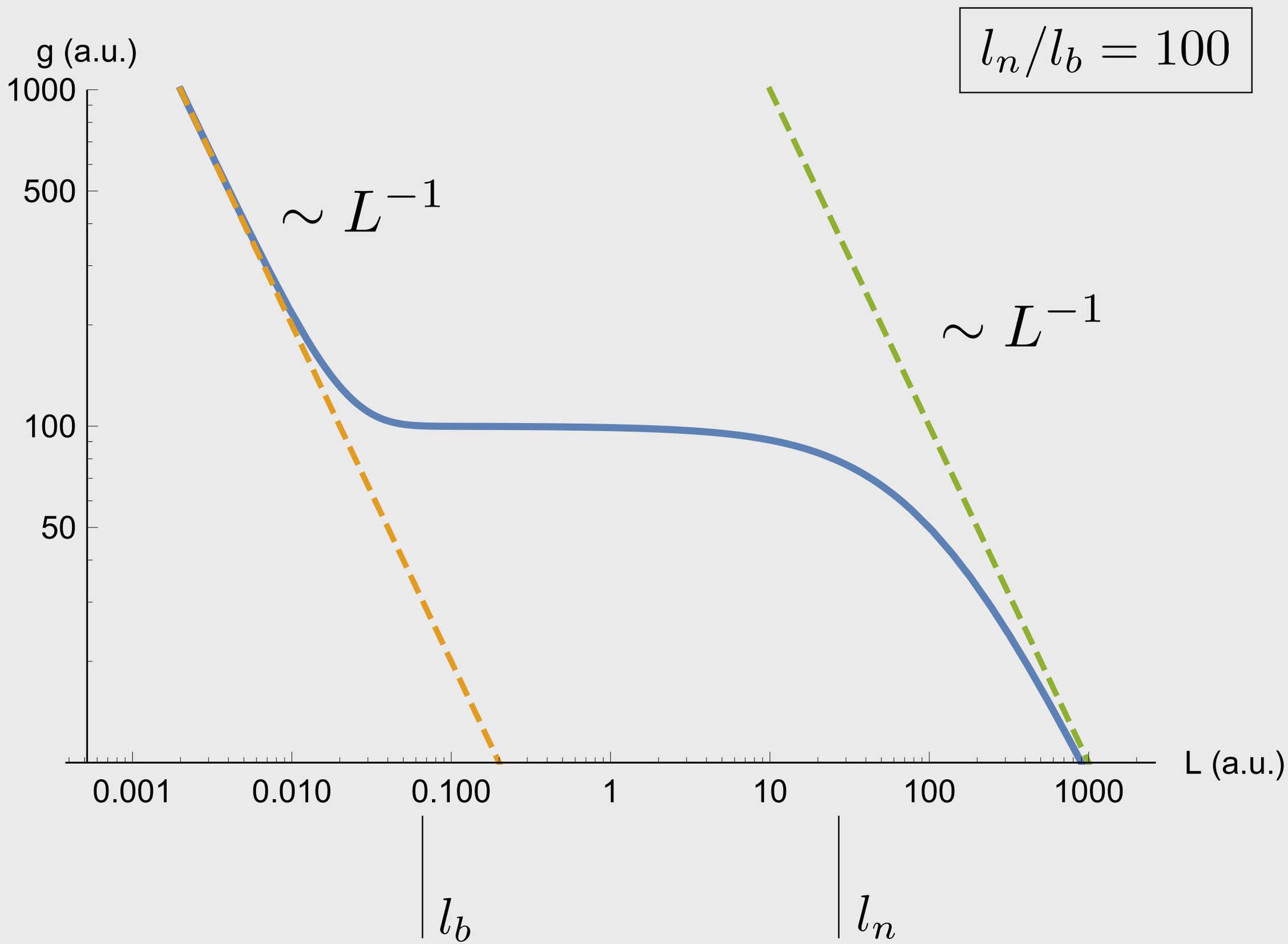
limits: diffusive transport at short (!!) length scales $L < l_b < l_n$

$$g = g_0 + \frac{B^2 \tau_a}{4\pi^2 L}$$

magnetoconductance



Ballistic conductance



summary

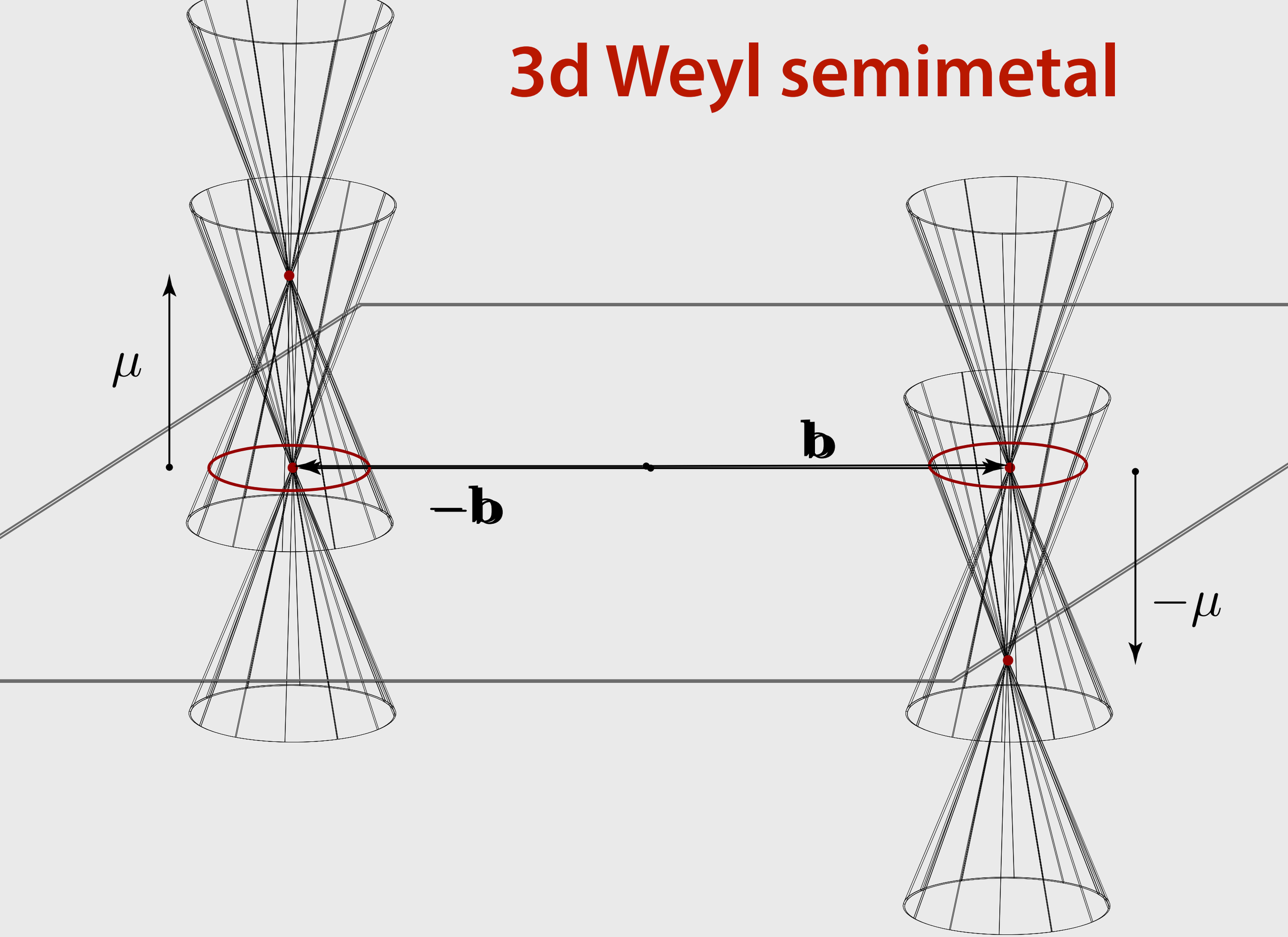
strong disorder phase is 3d Anderson metal with

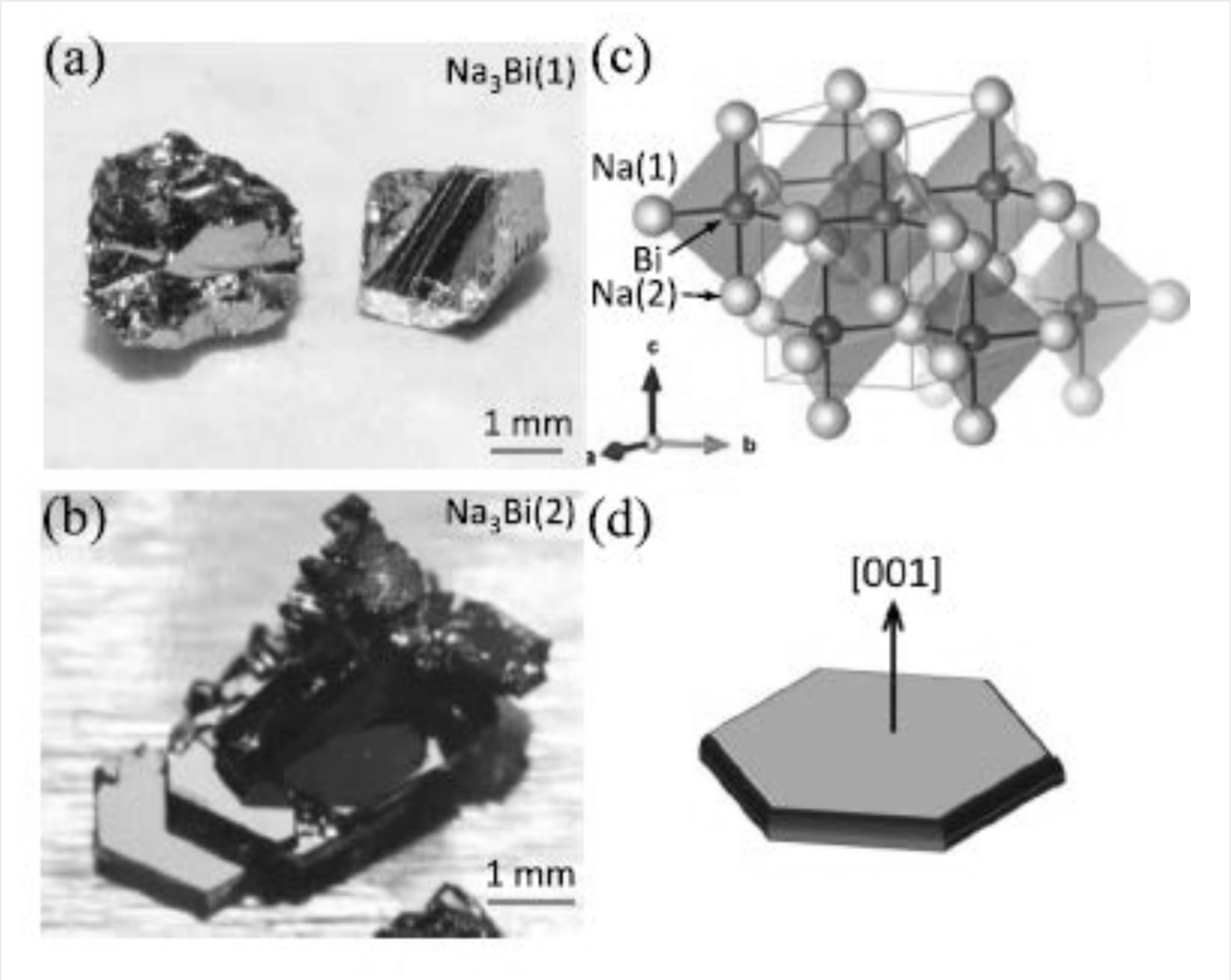
stable Hall response coefficient (AHE), and

topological term supporting bulk CME and

diffusion & drift stabilized by topology

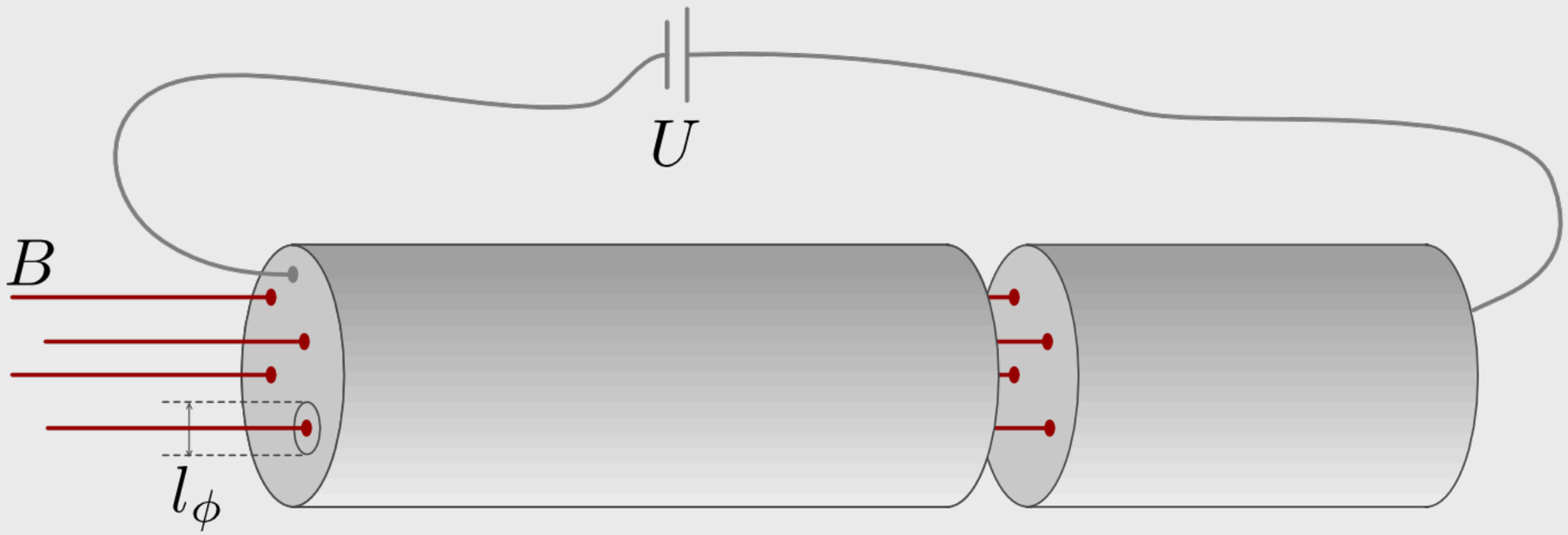
3d Weyl semimetal

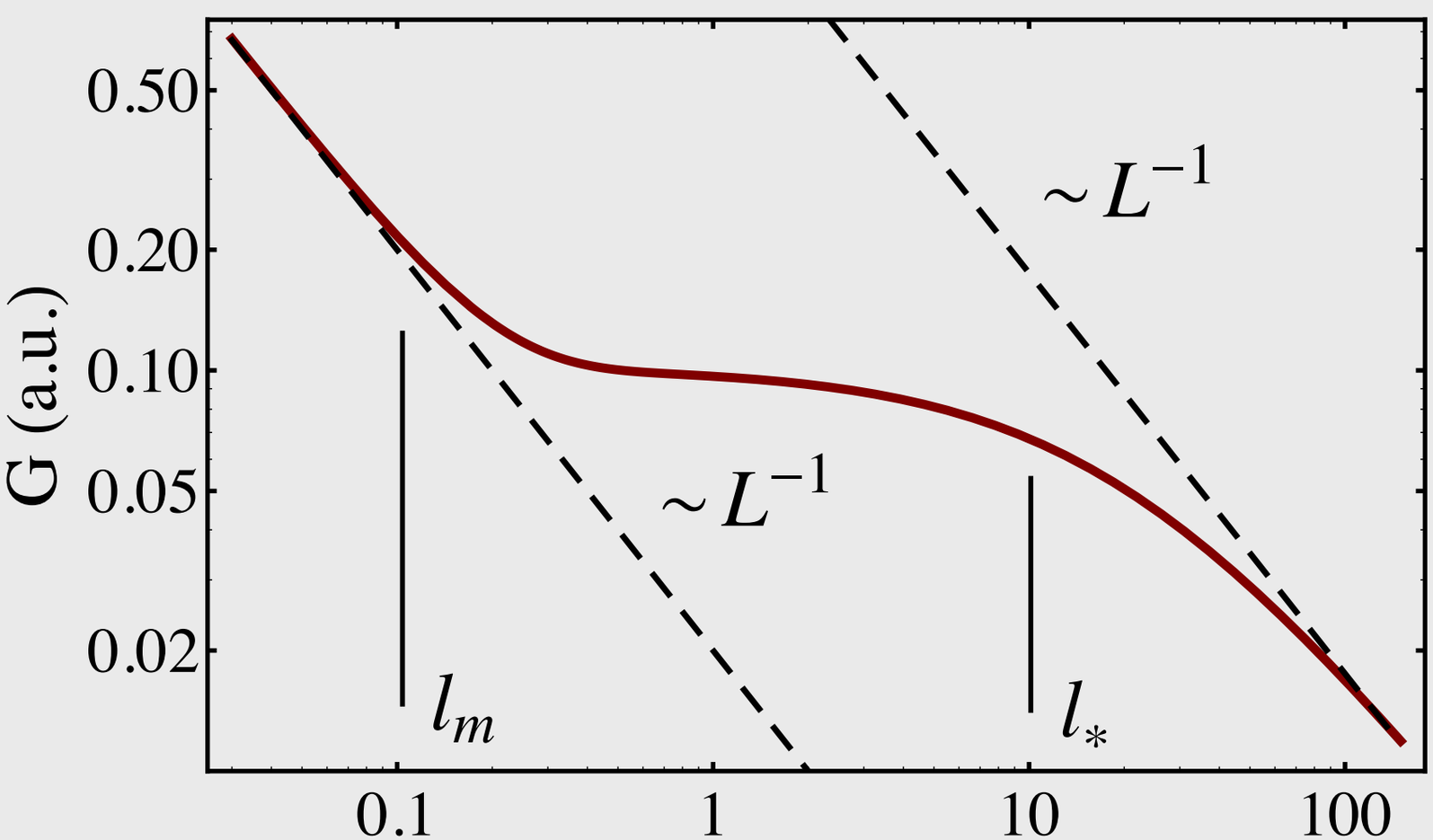




Xiong et al. 2015

quasi one-dimensional quantum transport





Summary

quantum criticality of topological insulators:
two parameter scaling

stabilization of topology by localization

boundary physics, quantum transport

2d Z insulators

ON LOCALIZATION IN THE THEORY OF THE QUANTIZED HALL EFFECT: A TWO-DIMENSIONAL REALIZATION OF THE θ-VACUUM

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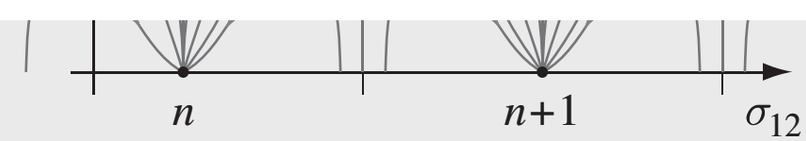
Schlumberger-Doll Research, PO Box 307, Ridgefield, CT 06877, USA

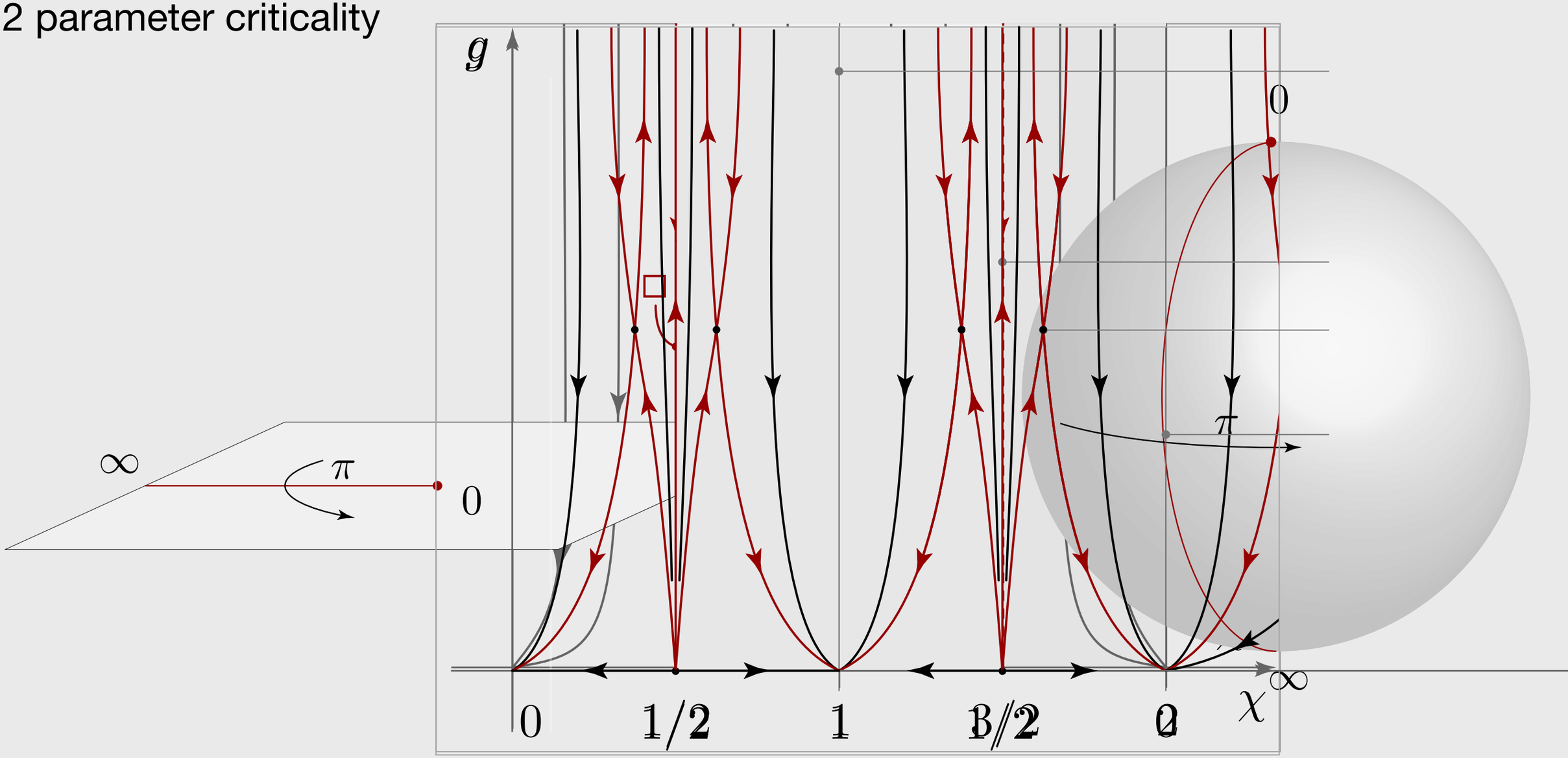
Received 28 July 1983
(Corrected version received 25 November 1983)

It is shown that the localization problem in the theory of the quantized Hall effect is governed by the zero-component grassmannian U($2m$) non-linear σ-model with a θ-term, a two-dimensional analogue of the θ-vacuum in Yang-Mills theory In this case, θ is to be interpreted as the "bare" value for the Hall conductivity, determined by an underlying non-critical theory A detailed derivation is presented starting from the replica method and a delta function distribution for the impurities



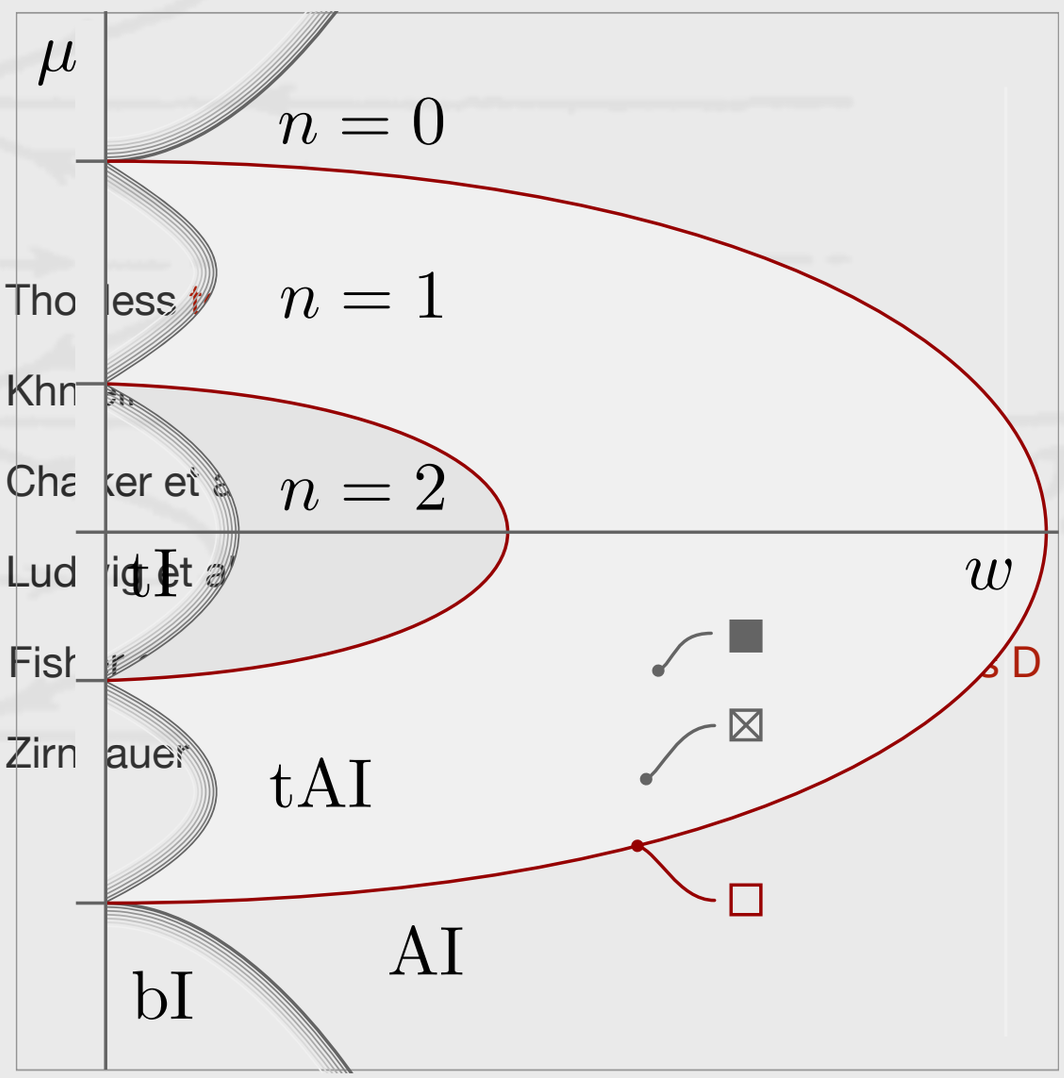
generalization to Z2

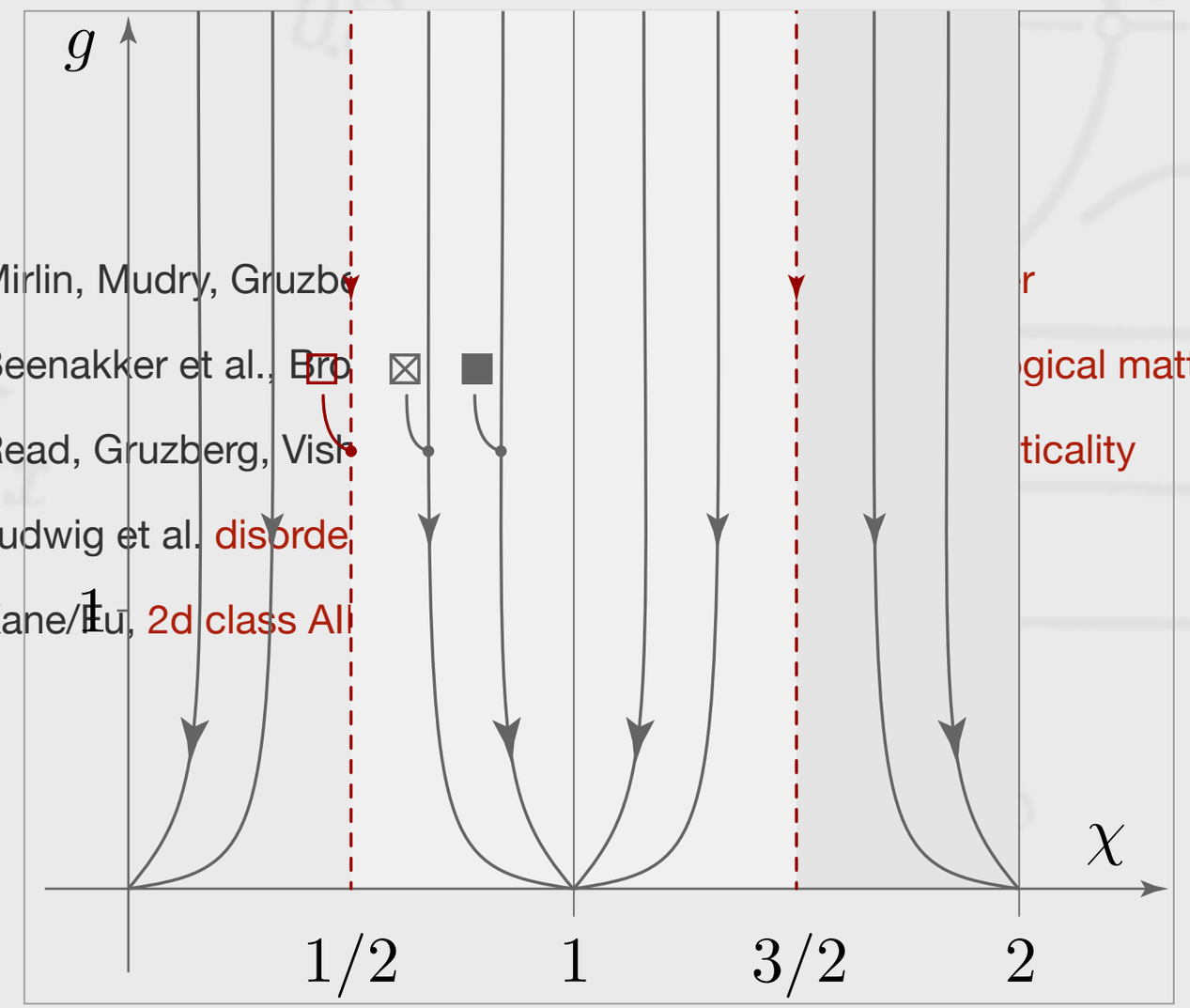
- case study: AII, d=2 (Kane & Fu, 2012)
  - system probed by topological point defects
  - field theory admits point-like excitations
  - theta-term —> fugactiy term
  - 2 parameter criticality



Summary

- quantum criticality of translationally non-invariant topological insulators
  - 2-parameter field theory
  - probed by continuous (Z) or point-like (Z2) topological sources
  - universal scaling
  - stabilization of topology by localization





Thouless
Khmelnitskii
Chalker et al.
Ludwig et al.
Fisher
Zirnbauer

Mirlin, Mudry, Gruzberg
Beenakker et al., Bro
Read, Gruzberg, Vish
Ludwig et al. disorde
Kane/Lu, 2d class AII

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