Transport in topological insulators

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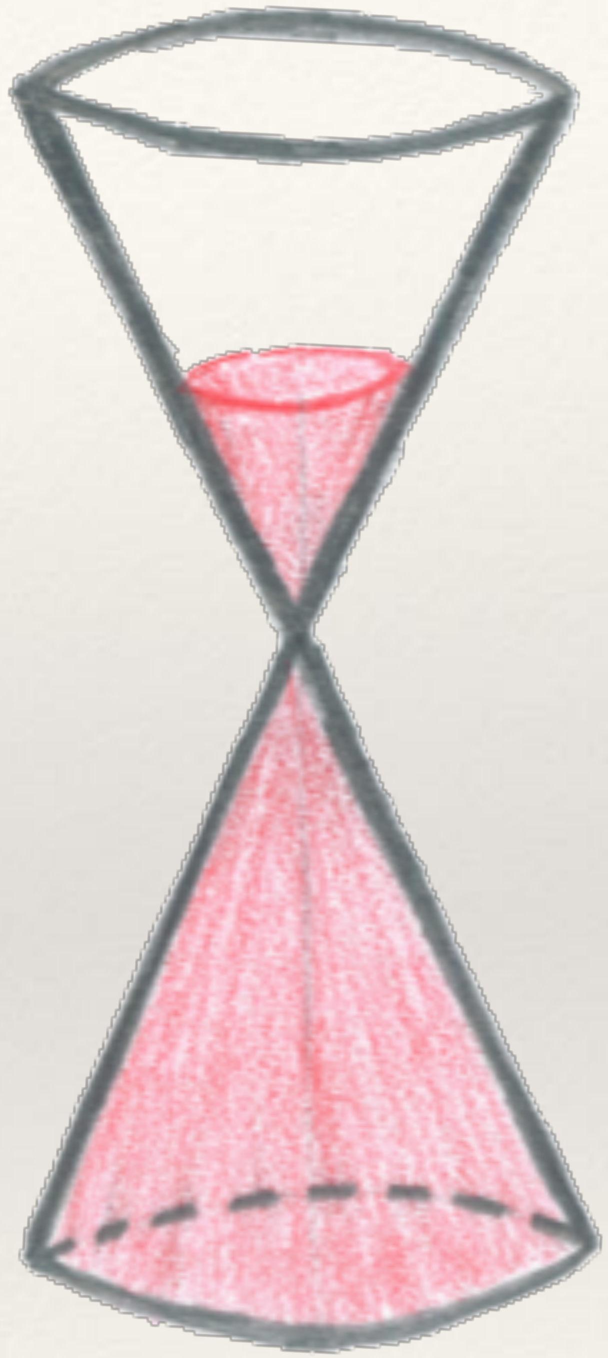
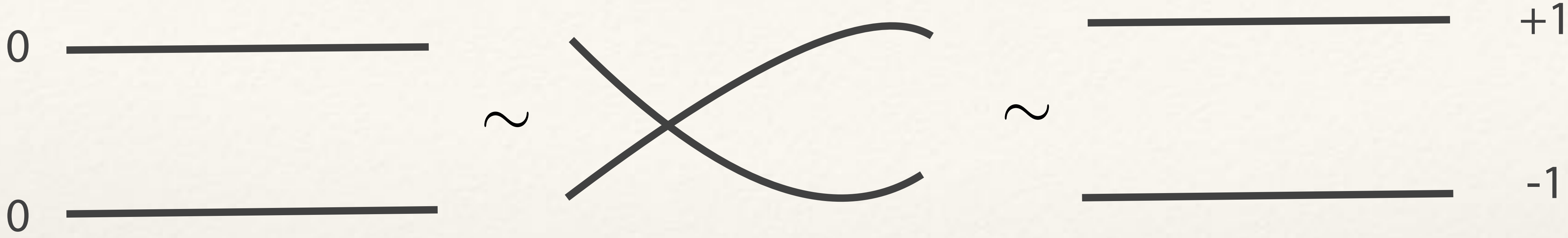
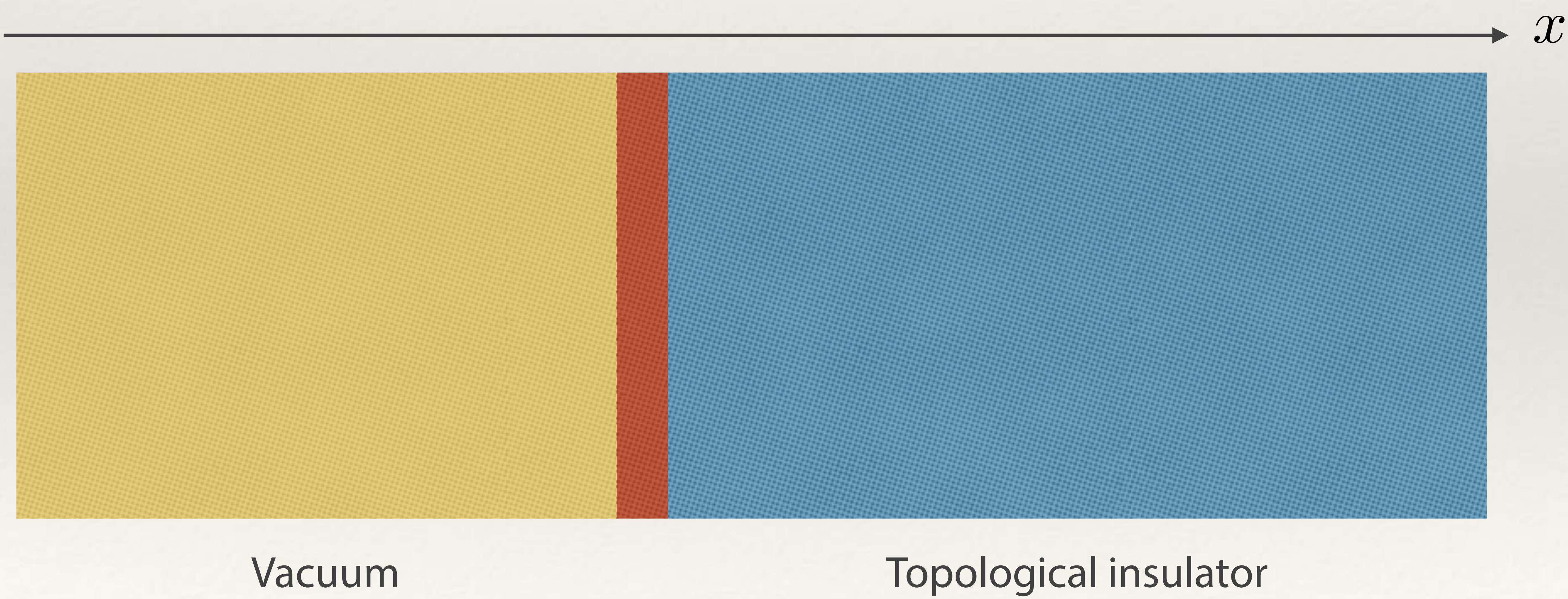


Brief intro

Topological insulators are bulk insulators with metallic surface





The periodic table of topological insulators

Cartan	T	C	S	H	NLσM	Name	d=0	d=1	d=2	d=3
A	0	0	0	U(N)	U(2n)/ U(n)xU(n)	Unitary	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	U(N+M)/ U(N)xU(M)	U(n)	Chiral Unitary	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	U(N)/O(N)	Sp(2n)/ Sp(n)xSp(n)	Orthogonal	$\mathbb{Z}$	0	0	0
BDI	+1	+1	1	O(N+M)/ O(N)xO(M)	U(2n)/Sp(2n)	Chiral orthogonal	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
D	0	+1	0	SO(2N)	O(2n)/U(n)	BdG	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	+1	1	SO(2N)/U(N)	O(2n)	BdG	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	U(2N)/Sp(2N)	O(2n)/ O(n)xO(n)	Symplectic	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	Sp(N+M)/ Sp(N)xSp(M)	U(2n)/O(2n)	Chiral symplectic	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	BdG	0	0	$2\mathbb{Z}$	0
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	BdG	0	0	0	$2\mathbb{Z}$

Ryu, Schneider, Furusaki, Ludwig NJP (2010), Ryu et al PRB, Kitaev …

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DIII	-1	+1	1	SO(2N)/U(N)	O(2n)	BdG	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
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C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	BdG	0	0	$2\mathbb{Z}$	0
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	BdG	0	0	0	$2\mathbb{Z}$

Ryu, Schneider, Furusaki, Ludwig NJP (2010), Ryu et al PRB, Kitaev …

Symmetries — Wigner's theorem

Any symmetry acts as a unitary or antiunitary transformation in Hilbert space

U unitary $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle$

T antiunitary $\langle T\psi|T\phi\rangle = \langle\psi|\phi\rangle^*$ (anti-linear!)

Can always write $T = UK$ with K complex conjugation

$T^2 = \pm 1$

Haake, Quantum Signatures of Chaos

Time reversal symmetry — Examples

spinless fermions (or integral spin):

$$T = K \qquad T^2 = K^2 = 1$$

$$T\mathbf{x}T^{-1} = \mathbf{x}$$ note: basis dependent!

$$T\mathbf{p}T^{-1} = K(-i\nabla)K = -\mathbf{p}$$

half-integral spin fermions:

$$T = i\sigma_y K \quad T^2 = (i\sigma_y K)(i\sigma_y K) = -\sigma_y^2 K^2 = -1$$

$$T\mathbf{x}T^{-1} = \mathbf{x}$$

$$T\mathbf{p}T^{-1} = K(-i\nabla)K = -\mathbf{p}$$

$$T\boldsymbol{\sigma}T^{-1} = -\boldsymbol{\sigma}$$

Haake, Quantum Signatures of Chaos

2D topological insulator

Topological insulators in the unitary class (A) — Quantum Hall effect

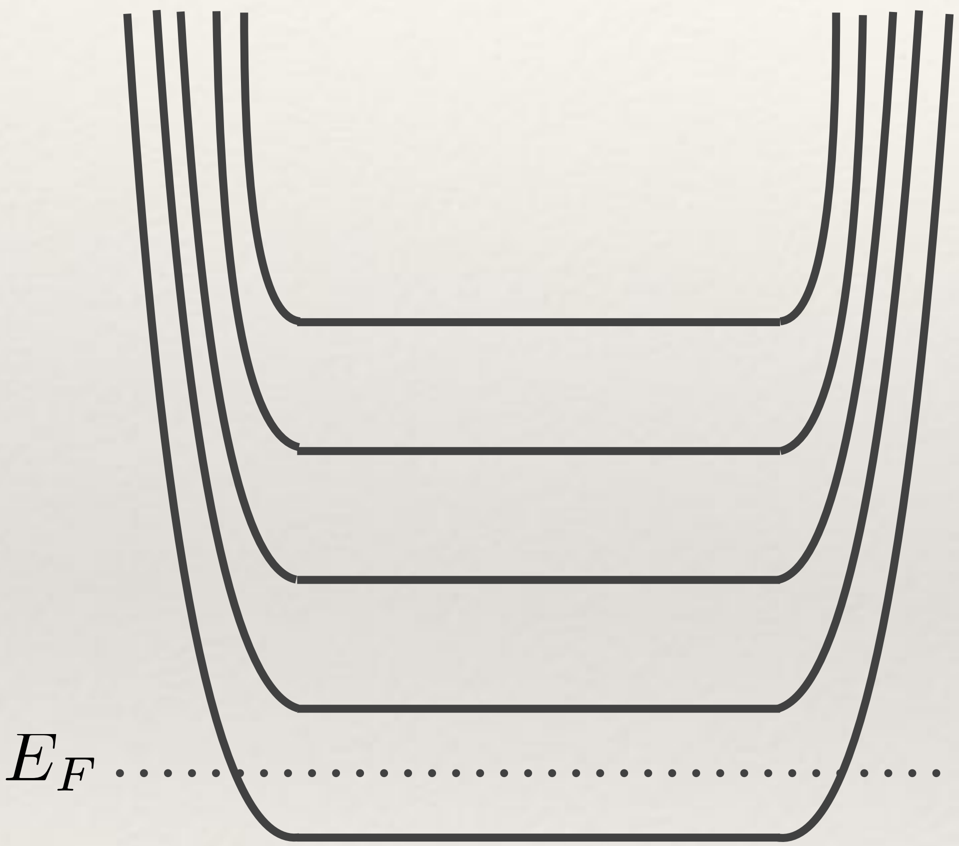
$$H = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m}$$

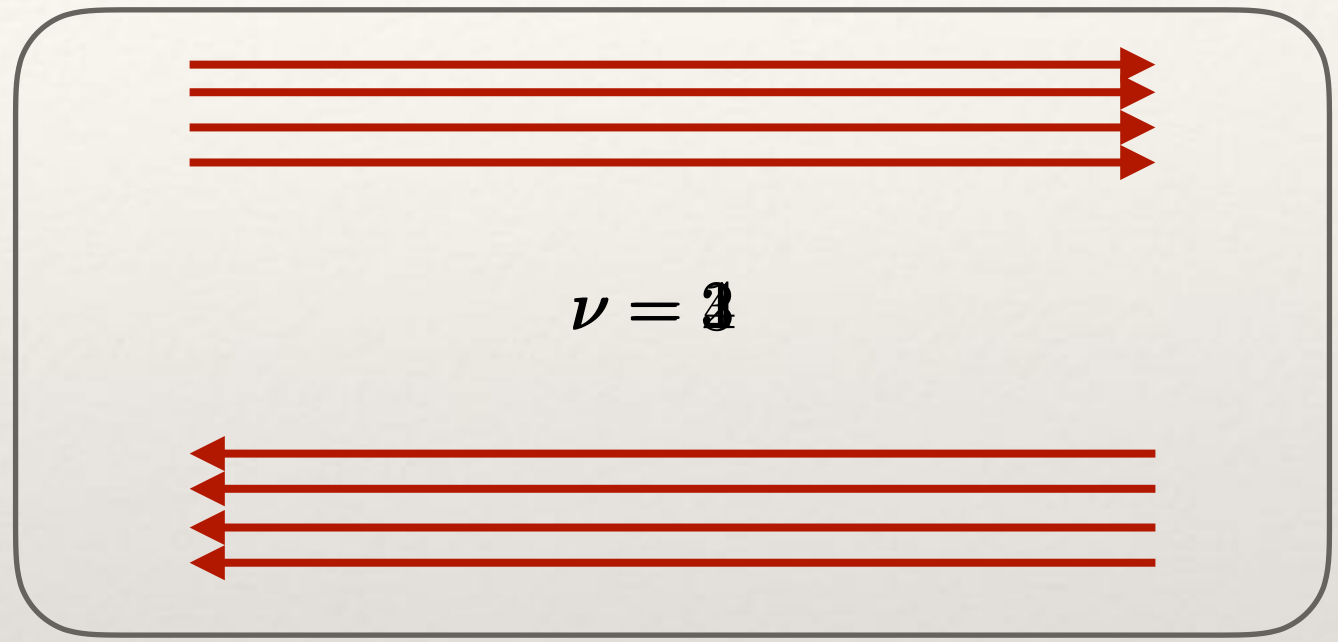
$$T = K$$

$$T\mathbf{p}T^{-1} = -\mathbf{p}$$

$$THT^{-1} = \frac{(-\mathbf{p} - e\mathbf{A})^2}{2m} \neq H$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$





$$\sigma_{xy} = \frac{e^2}{h}\nu \qquad \nu \in \mathbb{Z}$$

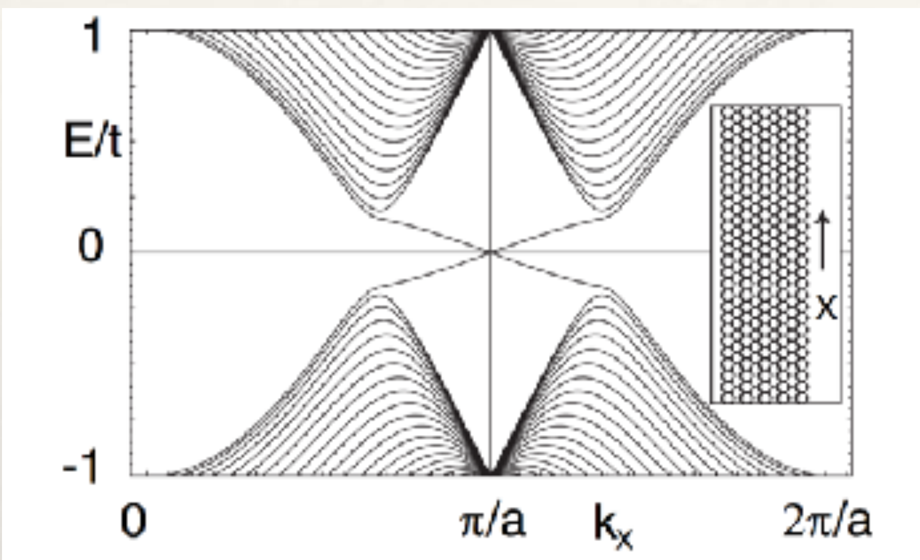
Landau levels

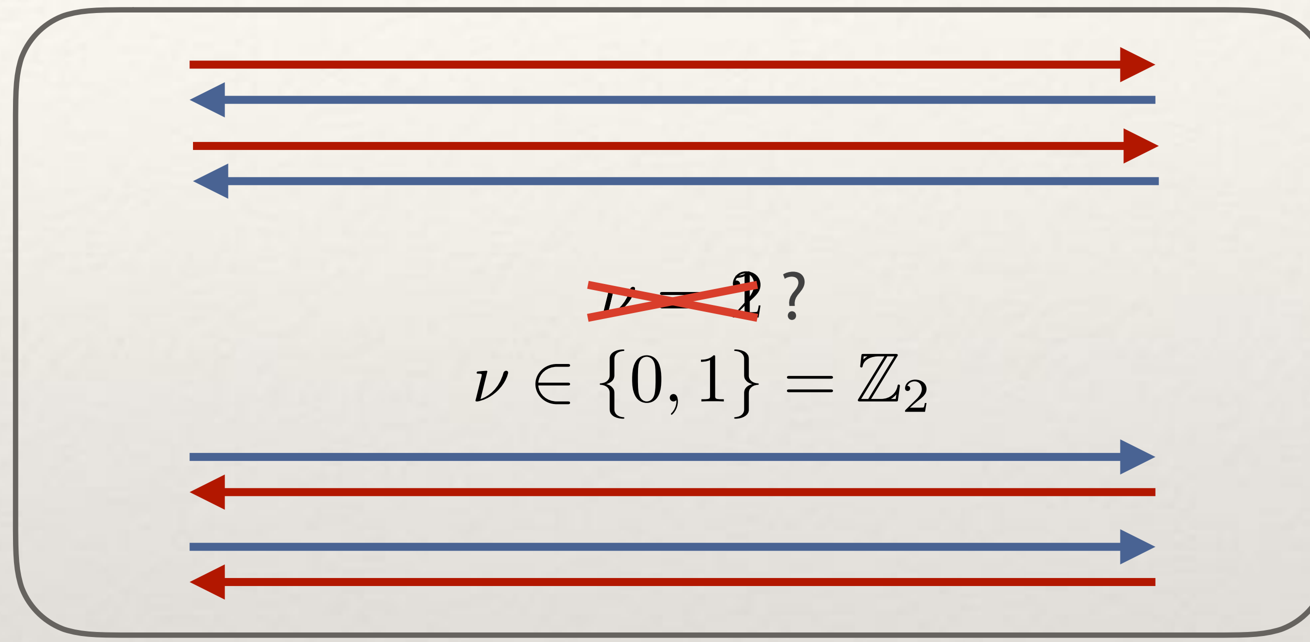
surface chiral — no backscattering

Topological insulators in the symplectic class (AII) — Quantum spin Hall effect

$$H = v_F(p_x\sigma_x\tau_z + p_y\sigma_y) + \Delta_{SO}\sigma_z\tau_z s_z + \lambda_R(\sigma_x\tau_z\sigma_y - \sigma_y s_x)$$

$$T = i\tau_x s_y K \qquad T^2 = -1 \qquad THT^{-1} = H$$





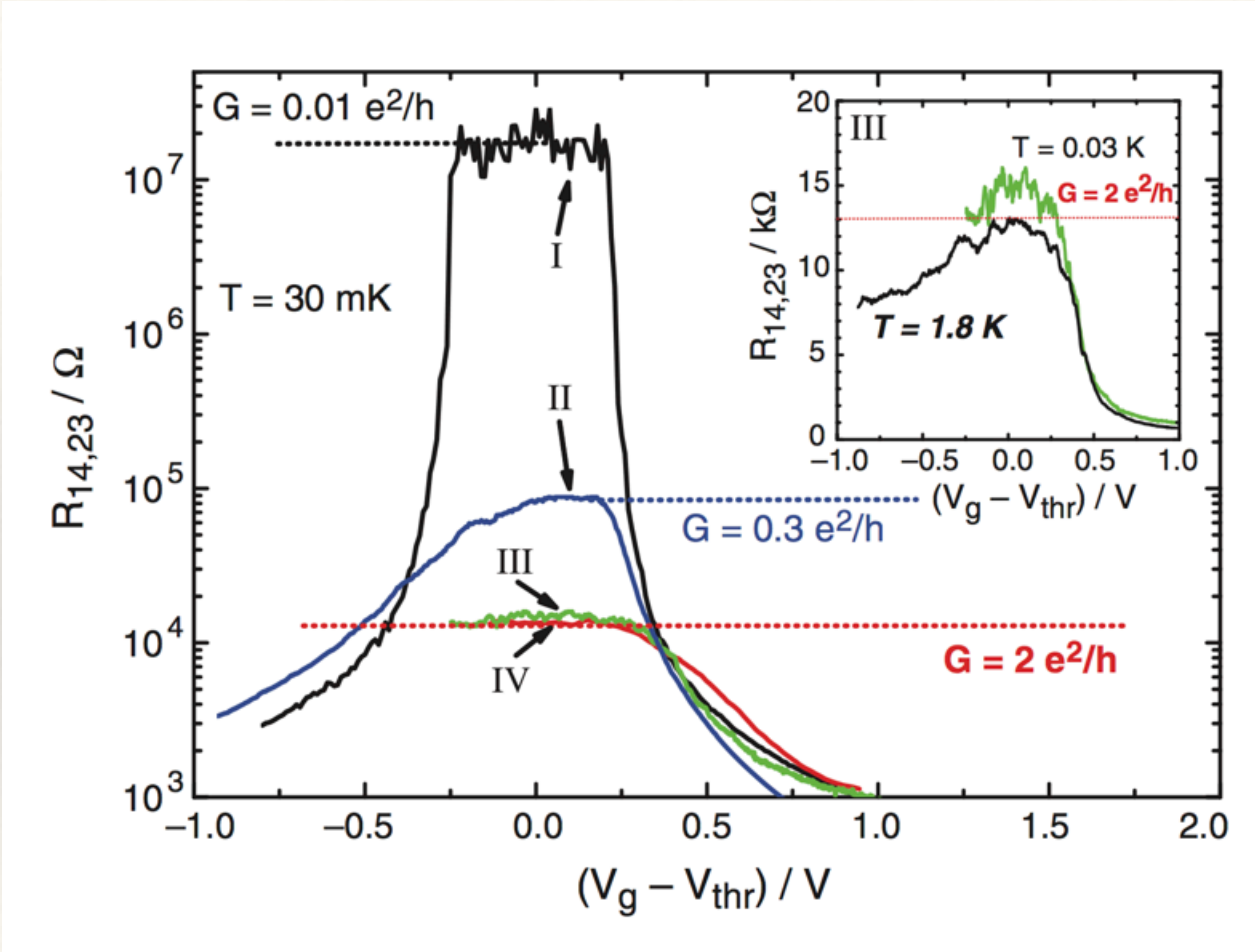
$$[T, V] = 0$$

$$\langle\psi|V|T\psi\rangle = \langle T\psi|TV|T\psi\rangle^* = \langle T\psi|VT|T\psi\rangle^* = -\langle T\psi|V|\psi\rangle^* = -\langle\psi|V|T\psi\rangle$$
$$= 0$$

Absence of backscattering $\qquad G = \frac{2e^2}{h}$

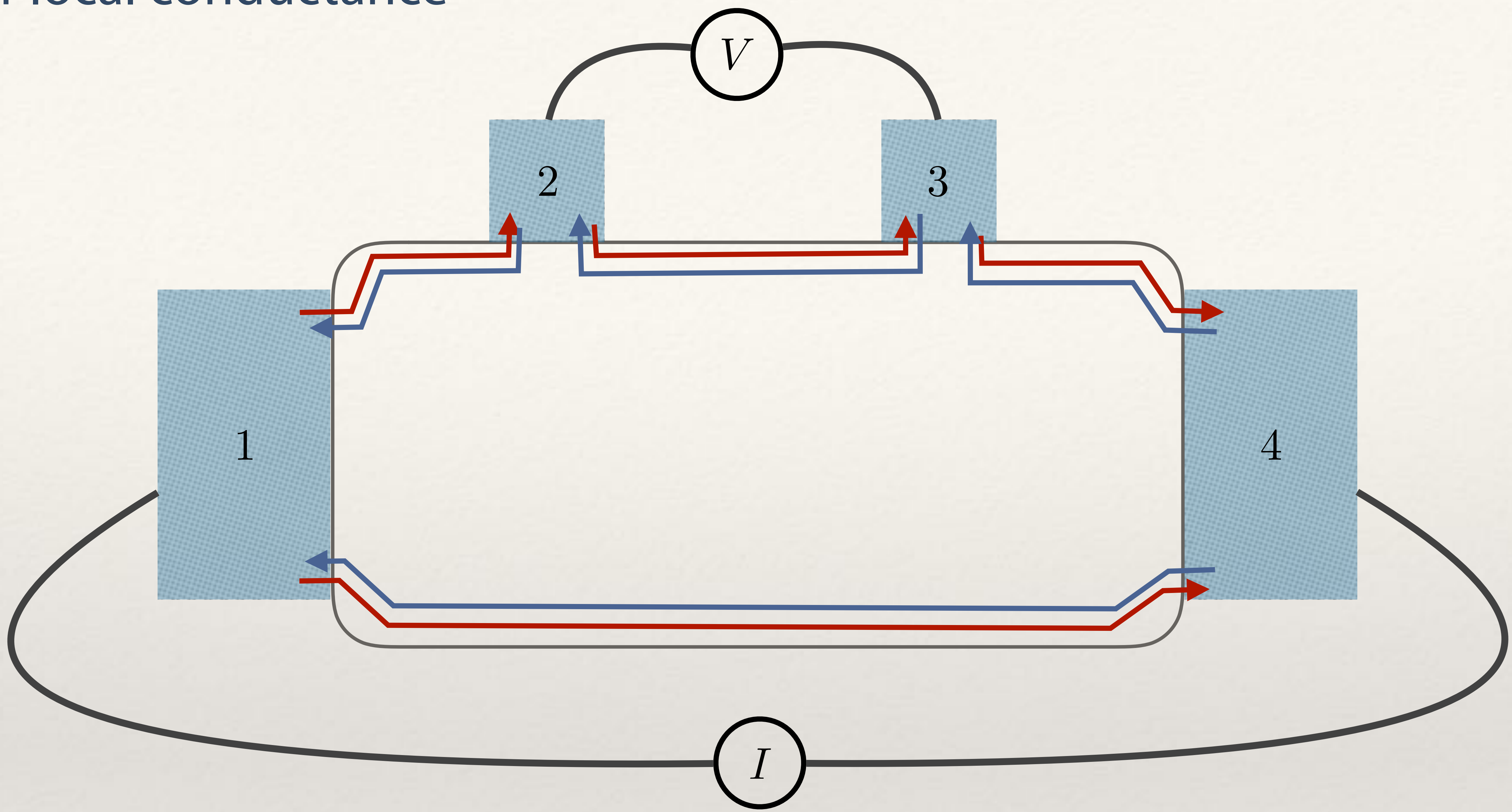
Kane and Mele, PRL 2005

Quantum Spin Hall effect in HgTe quantum wells



König et al, Science (2007)

Non-local conductance

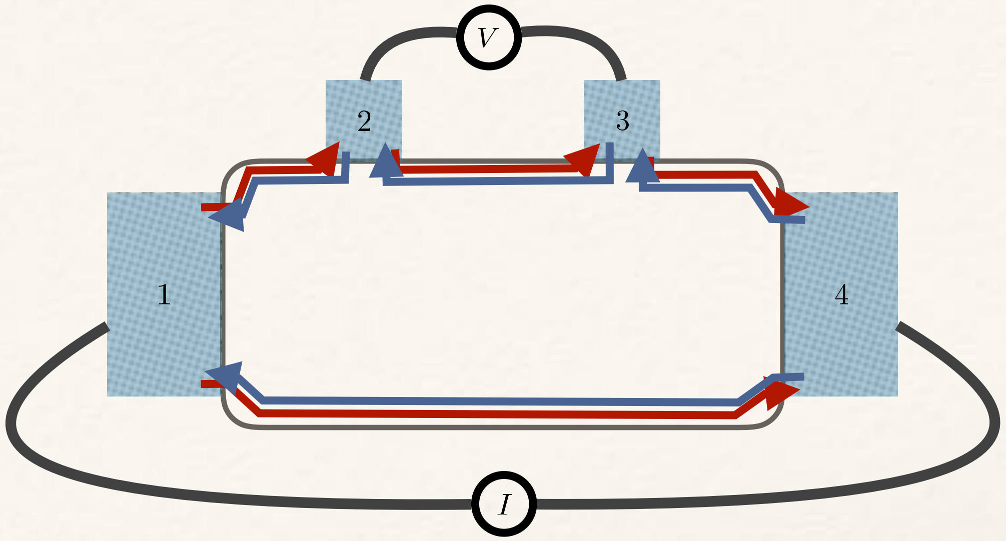


$$I_i = \frac{e^2}{h} \sum_j T_{ij}(V_i - V_j)$$

$I_1 = I = -I_4$ $\quad$ $I_2 = I_3 = 0$

$V_2 = 0$ $\quad$ $V_3 = V$

Non-local conductance



$$I = I_1 = (V_1 - V_2) + (V_1 - V_4) = 2V_1 - V_4$$

$$0 = I_2 = (V_2 - V_1) + (V_2 - V_3) = -V_1 - V$$

$$0 = I_3 = (V_3 - V_2) + (V_3 - V_4) = 2V - V_4$$

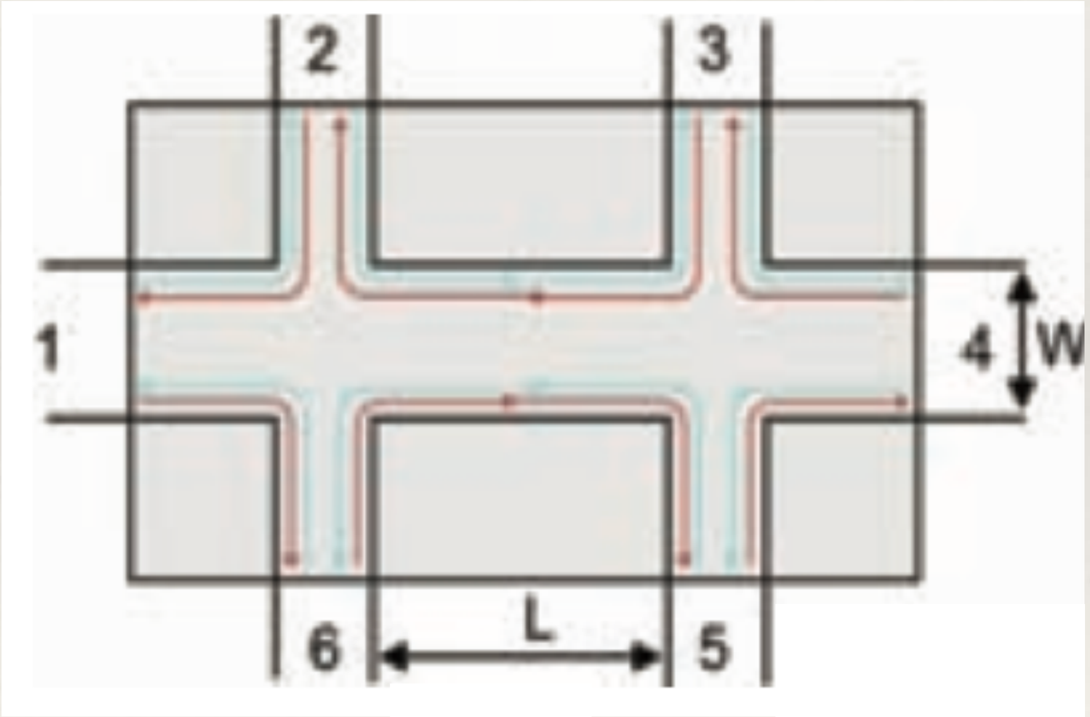
$$-I = I_4 = (V_4 - V_3) + (V_4 - V_1) = 2V_4 - V - V_1$$

$$\Rightarrow \quad V_1 = -V$$

$$V_4 = 2V$$

$$\Rightarrow \quad I = -4V$$

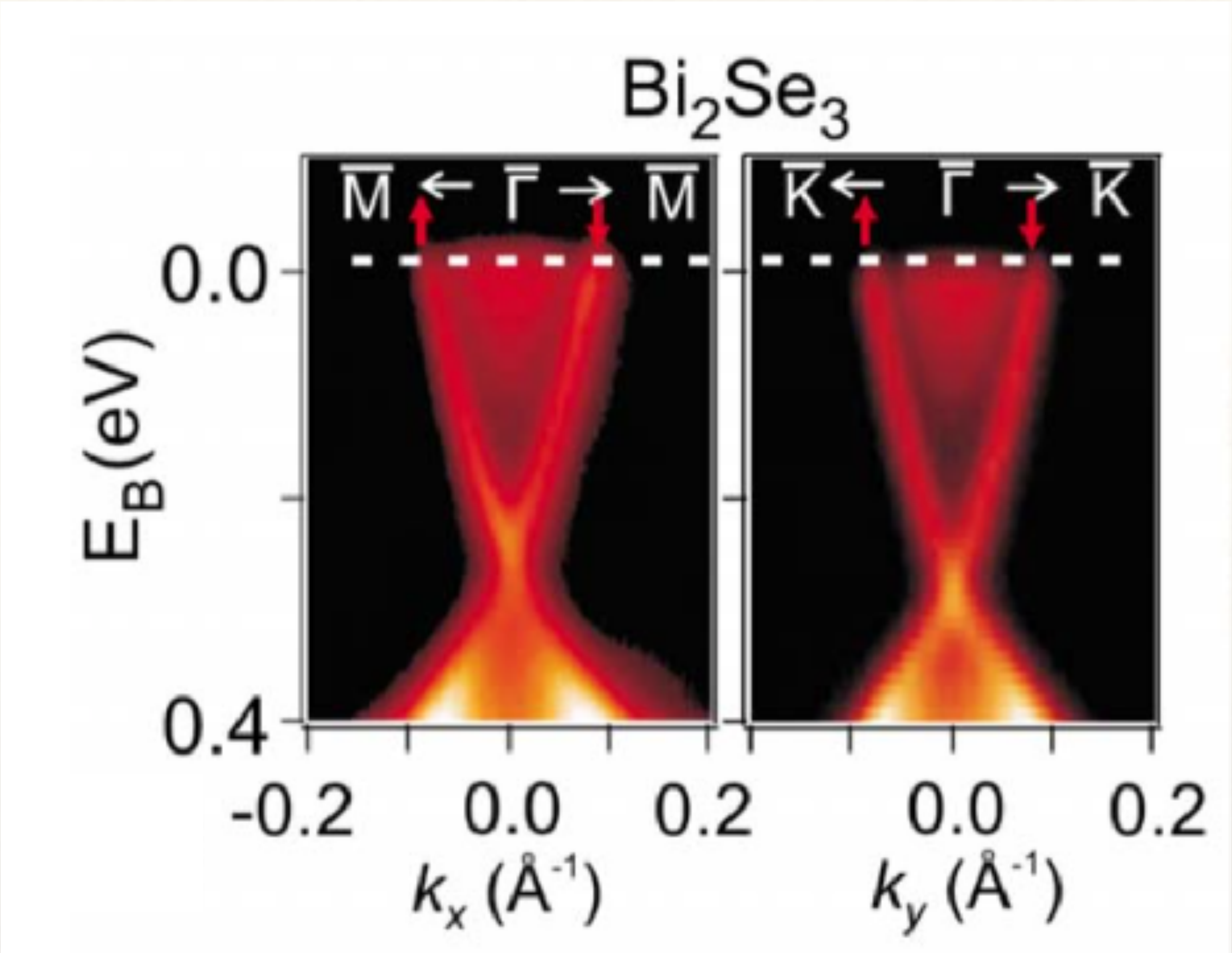
$$\Rightarrow \quad G = \frac{4e^2}{h}$$

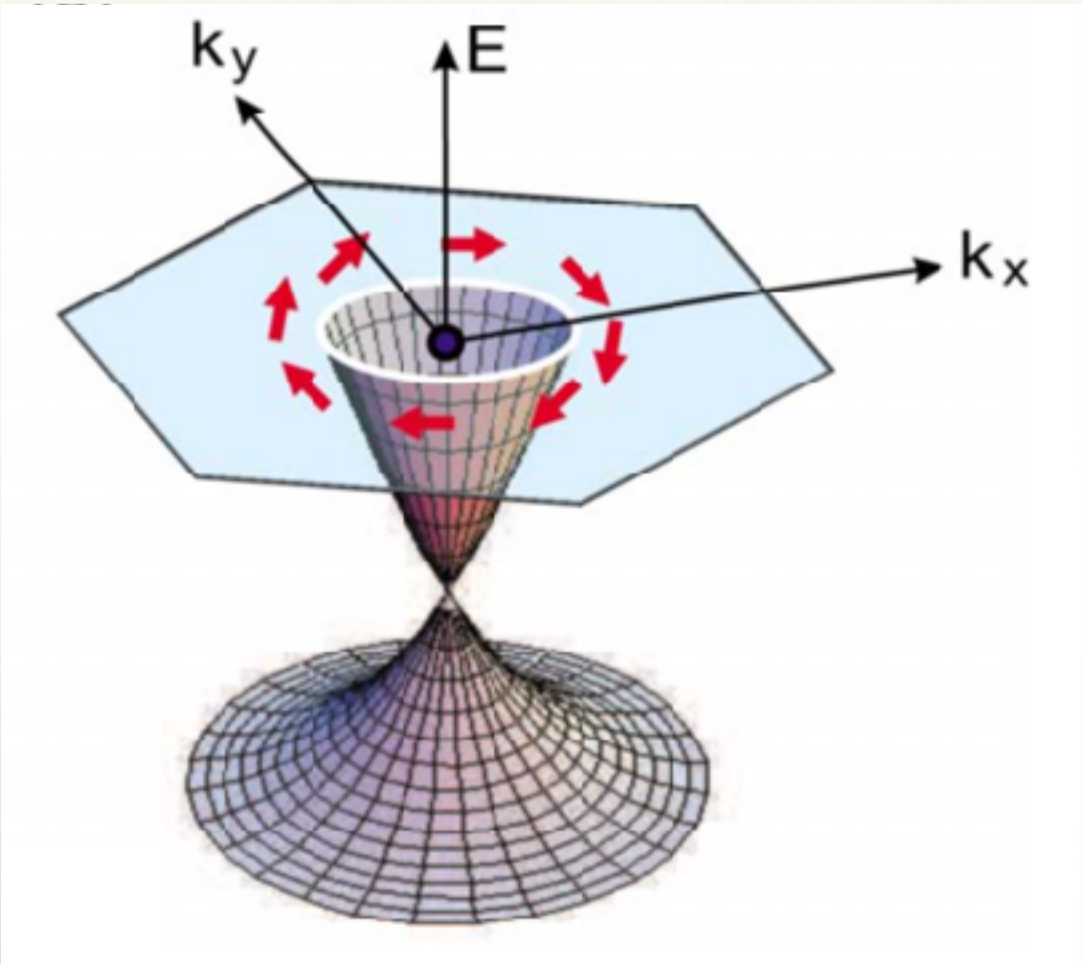


Ex. Find the conductance in the H-bar

3D topological insulator

3D topological insulators

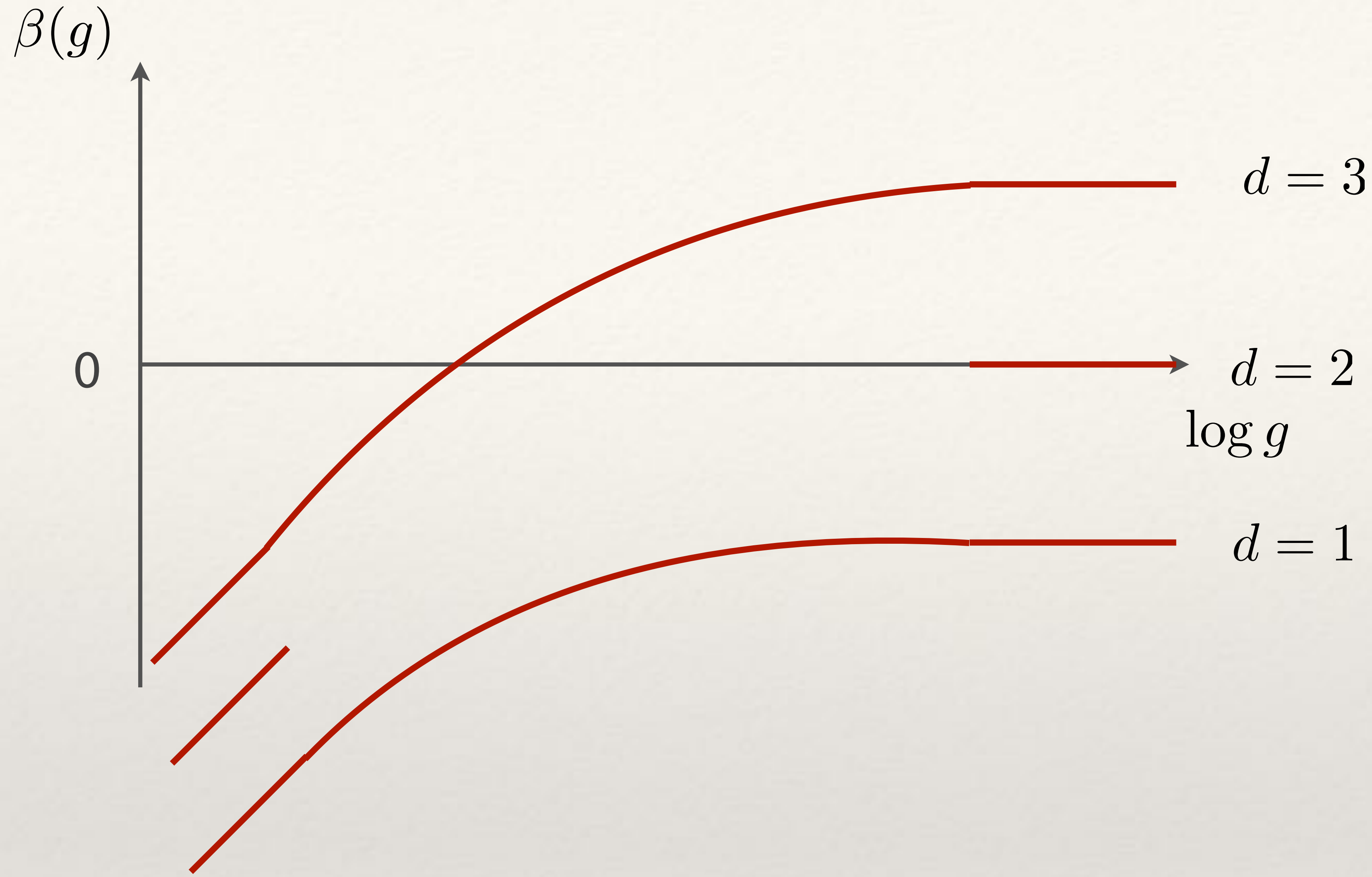




$$H_{\text{surface}} = \mathbf{p} \cdot \boldsymbol{\sigma}$$

Hasan and Kane RMP 2010

Scaling theory of localization



Drude

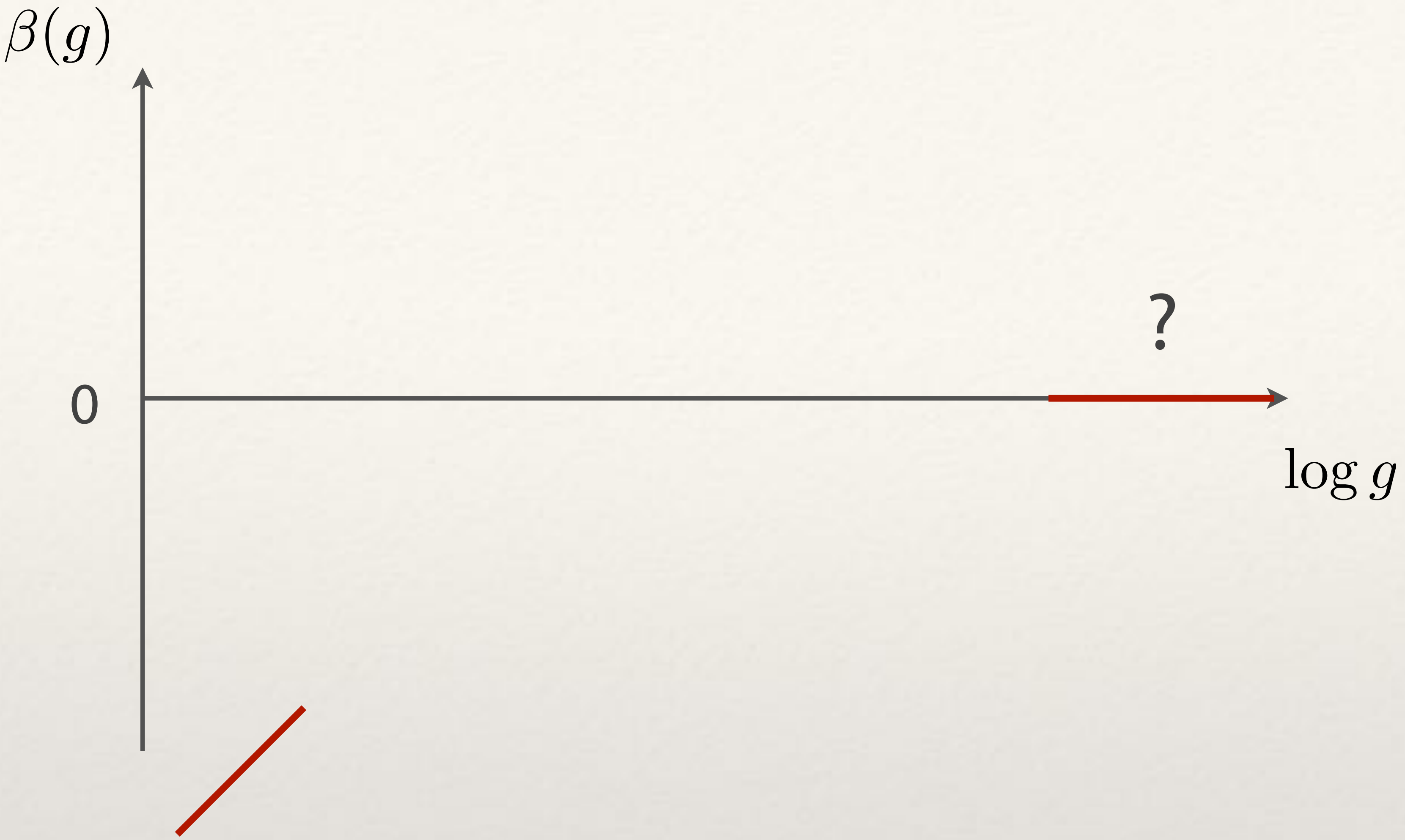
$$g = \sigma L^{d-2} \qquad \beta(g) = \frac{\partial \log g}{\partial \log L} = (d-2)$$

Anderson

$$g = g_0 e^{-L/\xi} \qquad \beta(g) = \log(g/g_0)$$

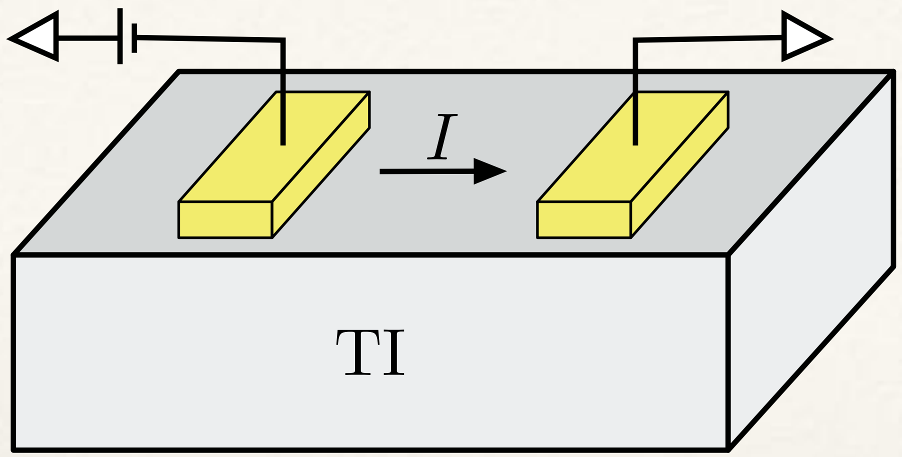
Abrahams, Anderson, Licciardello, and Ramakrishnan PRL 1979

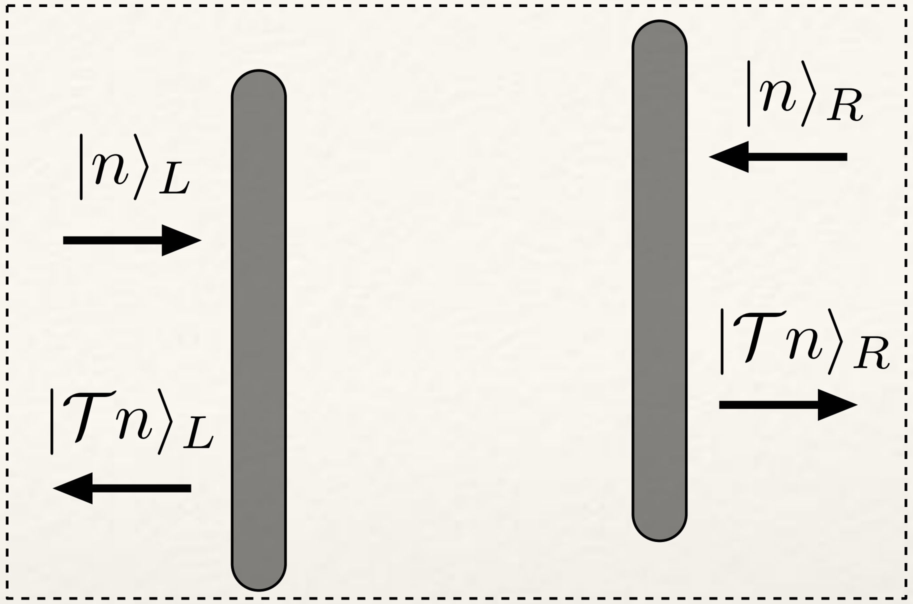
Scaling theory of localization



Abrahams, Anderson, Licciardello, and Ramakrishnan PRL 1979

Conductance as transmission (Landauer)





$$|\psi\rangle = \begin{cases} \sum_n c_{n,\mathrm{L}} |n\rangle_\mathrm{L} + d_{n,\mathrm{L}} |\mathcal{T}n\rangle_\mathrm{L}\,, & x \leqslant 0, \\ \sum_n c_{n,\mathrm{R}} |n\rangle_\mathrm{R} + d_{n,\mathrm{R}} |\mathcal{T}n\rangle_\mathrm{R}\,, & x \geqslant L, \\ |\Psi\rangle\,, & 0 \leqslant x \leqslant L. \end{cases}$$

$$\begin{pmatrix} d_\mathrm{L} \\ d_\mathrm{R} \end{pmatrix} = S \begin{pmatrix} c_\mathrm{L} \\ c_\mathrm{R} \end{pmatrix}$$

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

$$G = \frac{e^2}{h} \mathrm{Tr}\; t^\dagger t = \frac{e^2}{h} \mathrm{Tr}\; (1 - r^\dagger r)$$

Weak (anti) localization

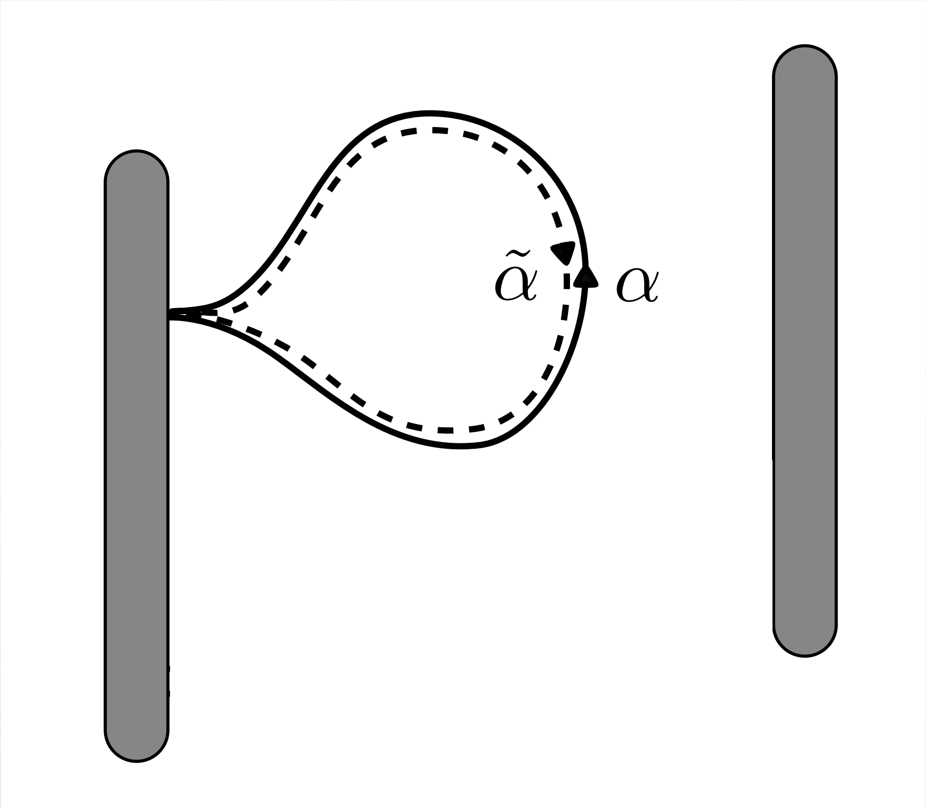
$$r_{nm} = \sum_{\alpha} A_\alpha e^{iS_\alpha/\hbar}$$

$$g = N - \sum_{\alpha,\alpha'} A_\alpha A^*_{\alpha'} e^{i(S_\alpha - S_{\alpha'})/\hbar}$$

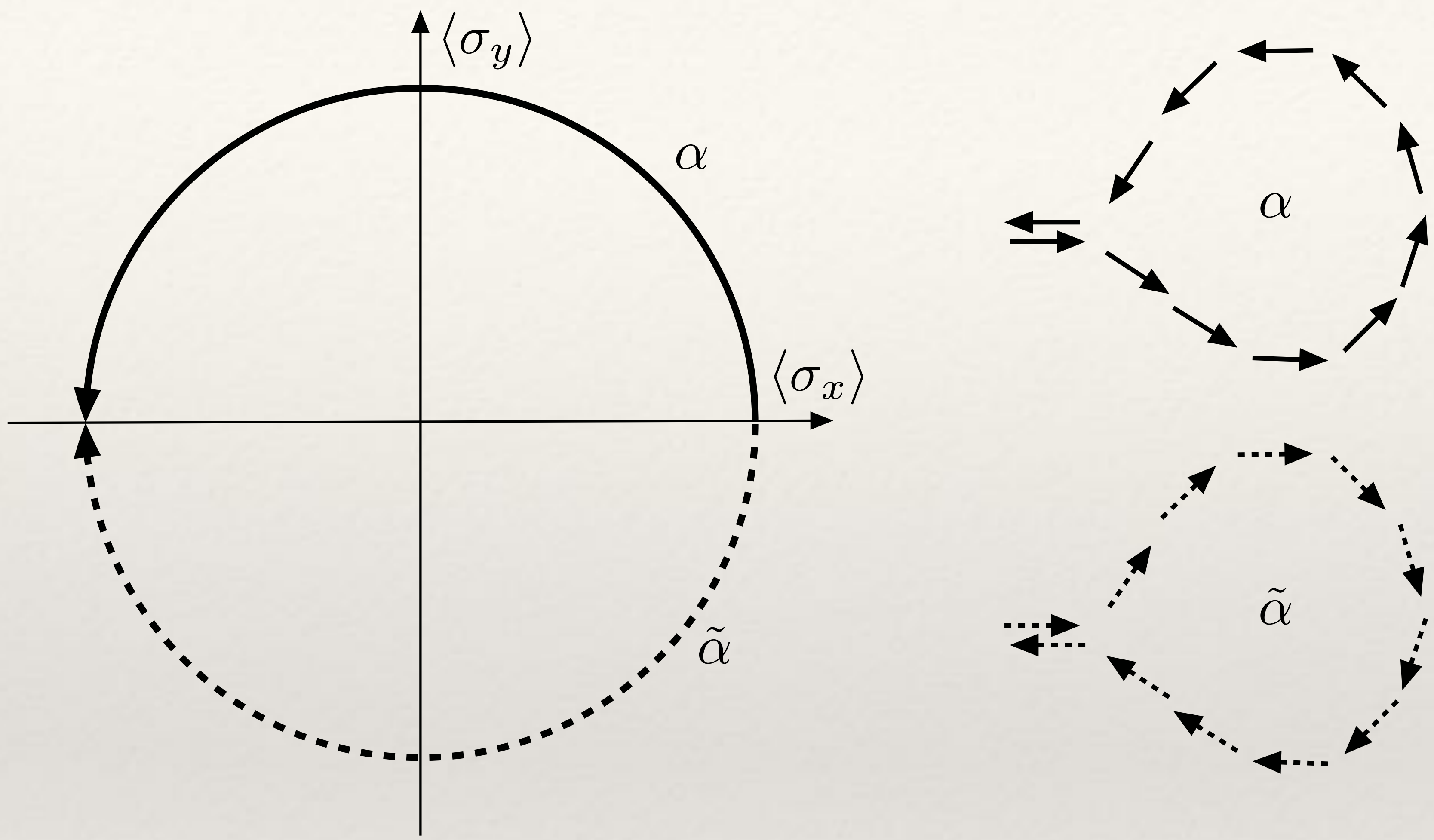
$$g_{\rm cl} = N - \sum_{\alpha} |A_\alpha|^2$$

$$\tilde{\alpha} = \mathcal{T}\alpha \quad \Rightarrow \quad S_{\tilde{\alpha}} = S_\alpha$$

$$|A_\alpha|^2 + |A_{\tilde{\alpha}}|^2 + 2\Re(A_\alpha A^*_{\tilde{\alpha}}) = ?$$



Spin-momentum locking and Berry's phase



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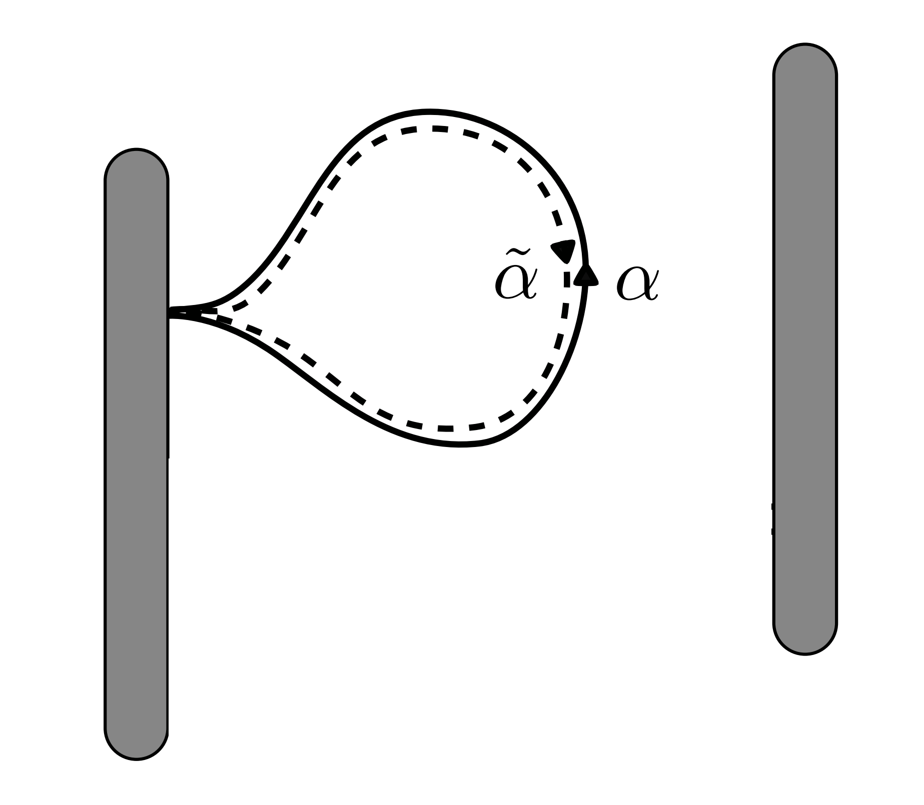
$$g_{\mathrm{cl}} = N - \sum_{\alpha} |A_\alpha|^2$$

$$\tilde{\alpha} = \mathcal{T}\alpha \quad \Rightarrow \quad S_{\tilde{\alpha}} = S_\alpha$$

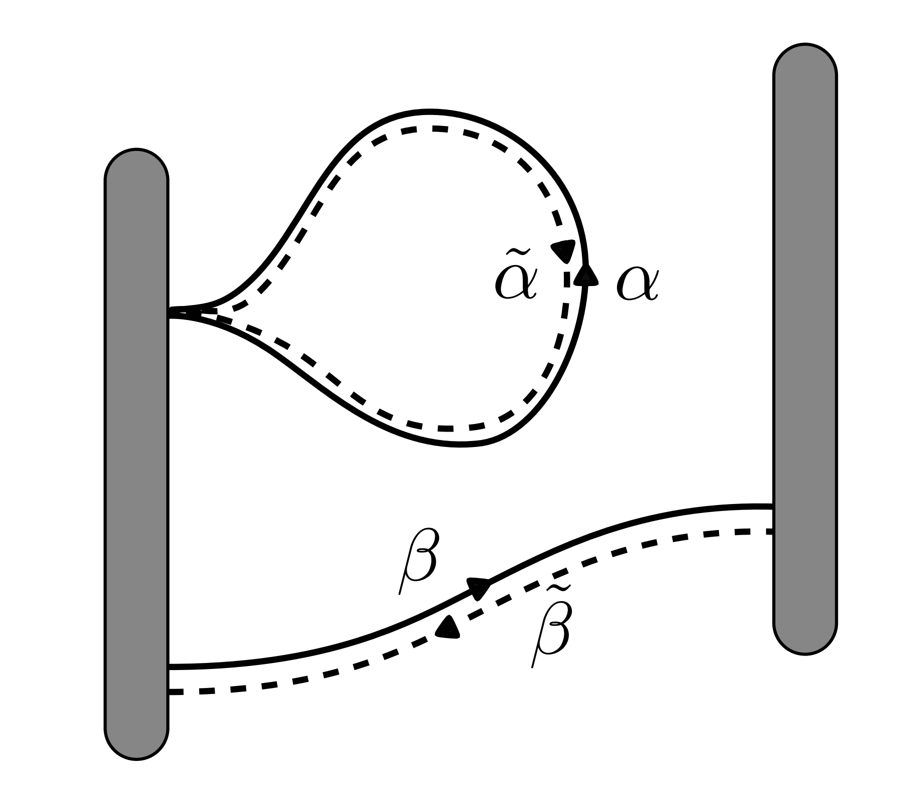
$$|A_\alpha|^2 + |A_{\tilde{\alpha}}|^2 + 2\Re(A_\alpha A^*_{\tilde{\alpha}}) = \begin{cases} 4|A_\alpha|^2, & T^2 = 1 \\ 0, & T^2 = -1 \end{cases}$$

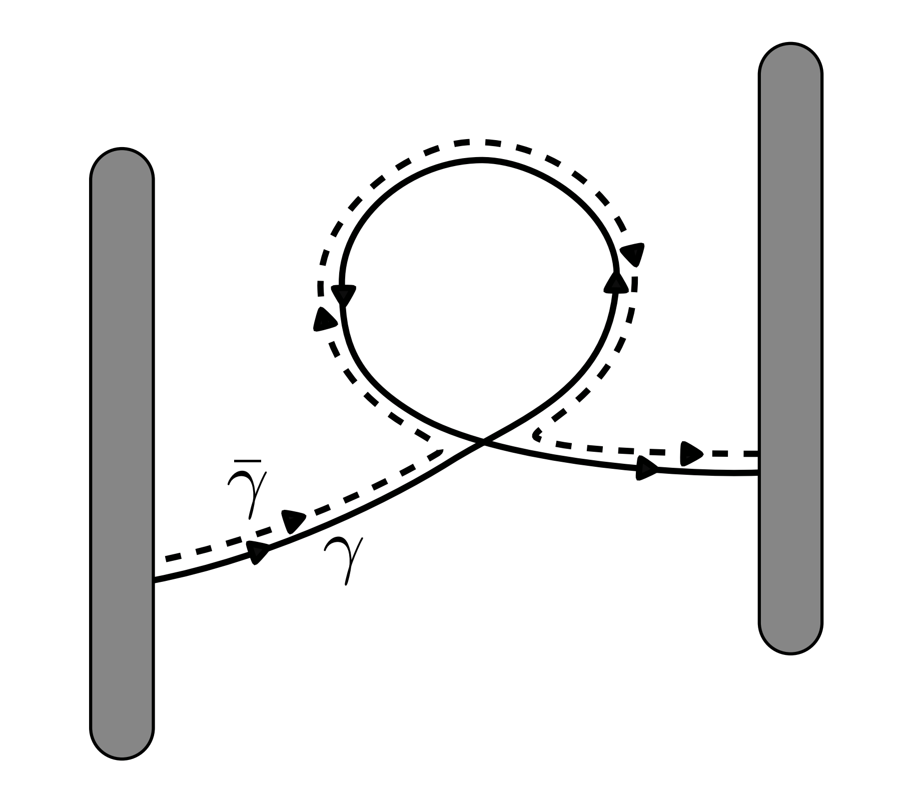


Weak (anti) localization

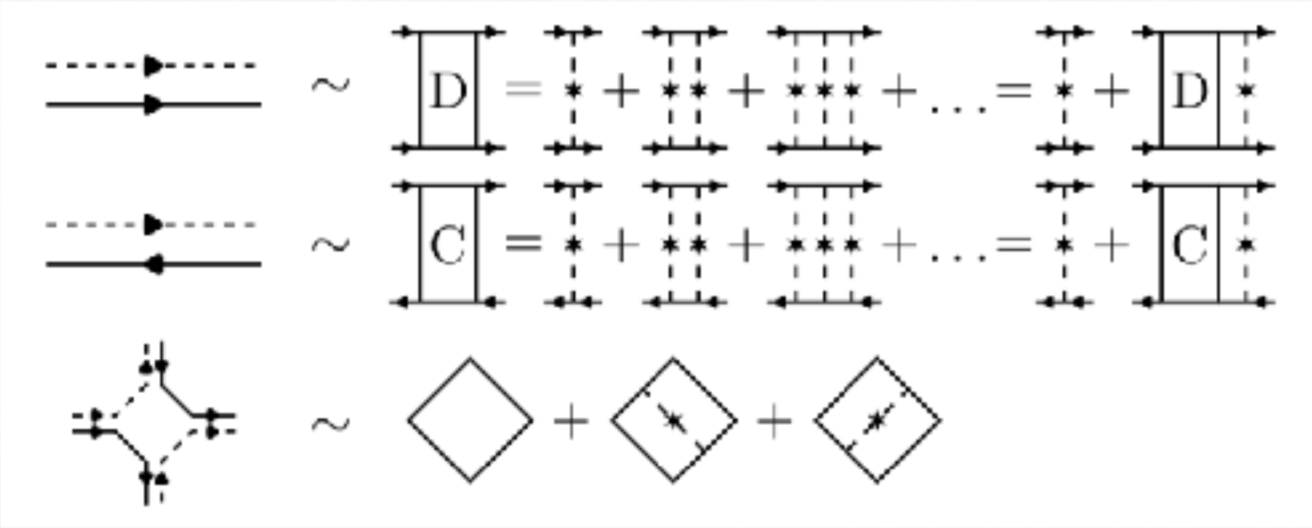
$$t_{nm} = \sum_{\beta} A_{\beta} e^{iS_{\beta}/\hbar}$$

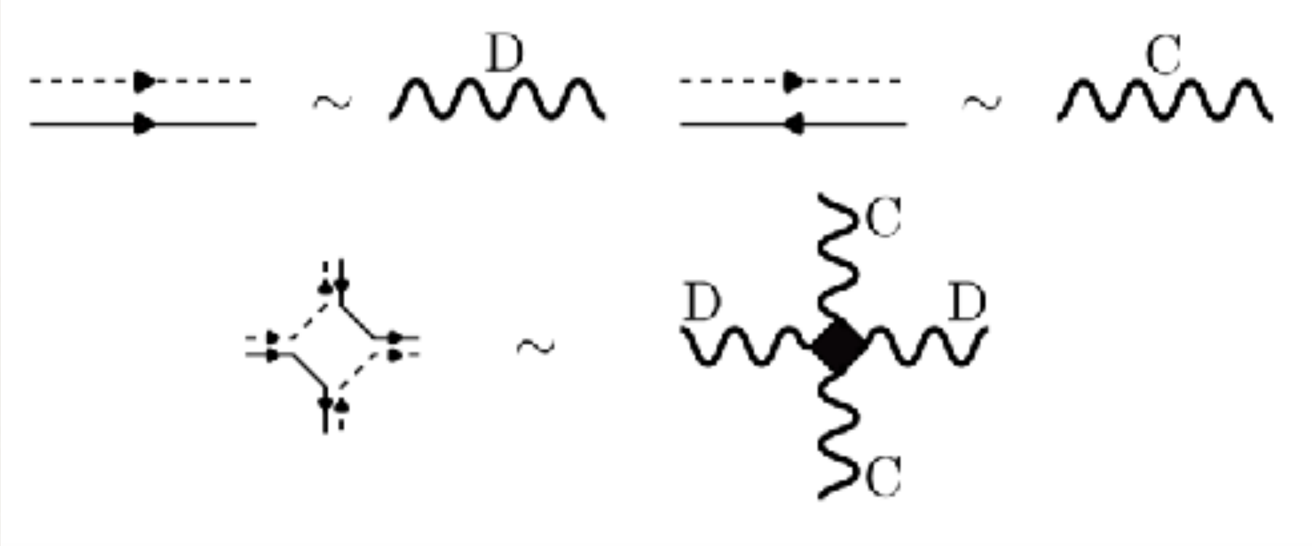
$$g = \sum_{\beta,\beta'} A_{\beta} A^{*}_{\beta'} e^{i(S_{\beta}-S_{\beta'})/\hbar}$$

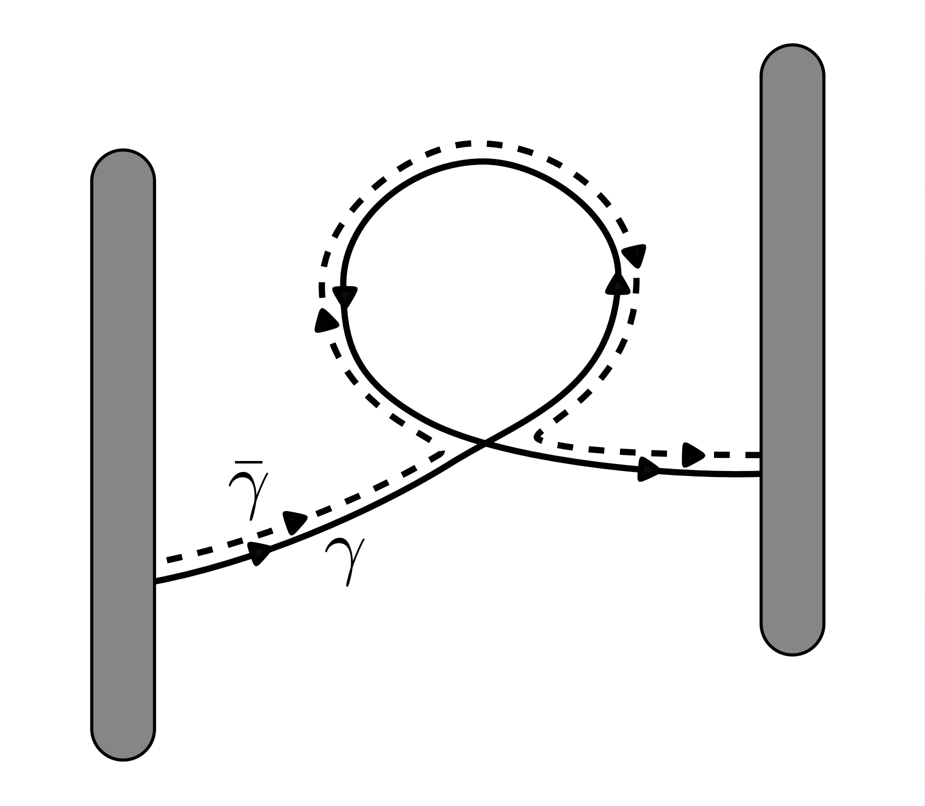




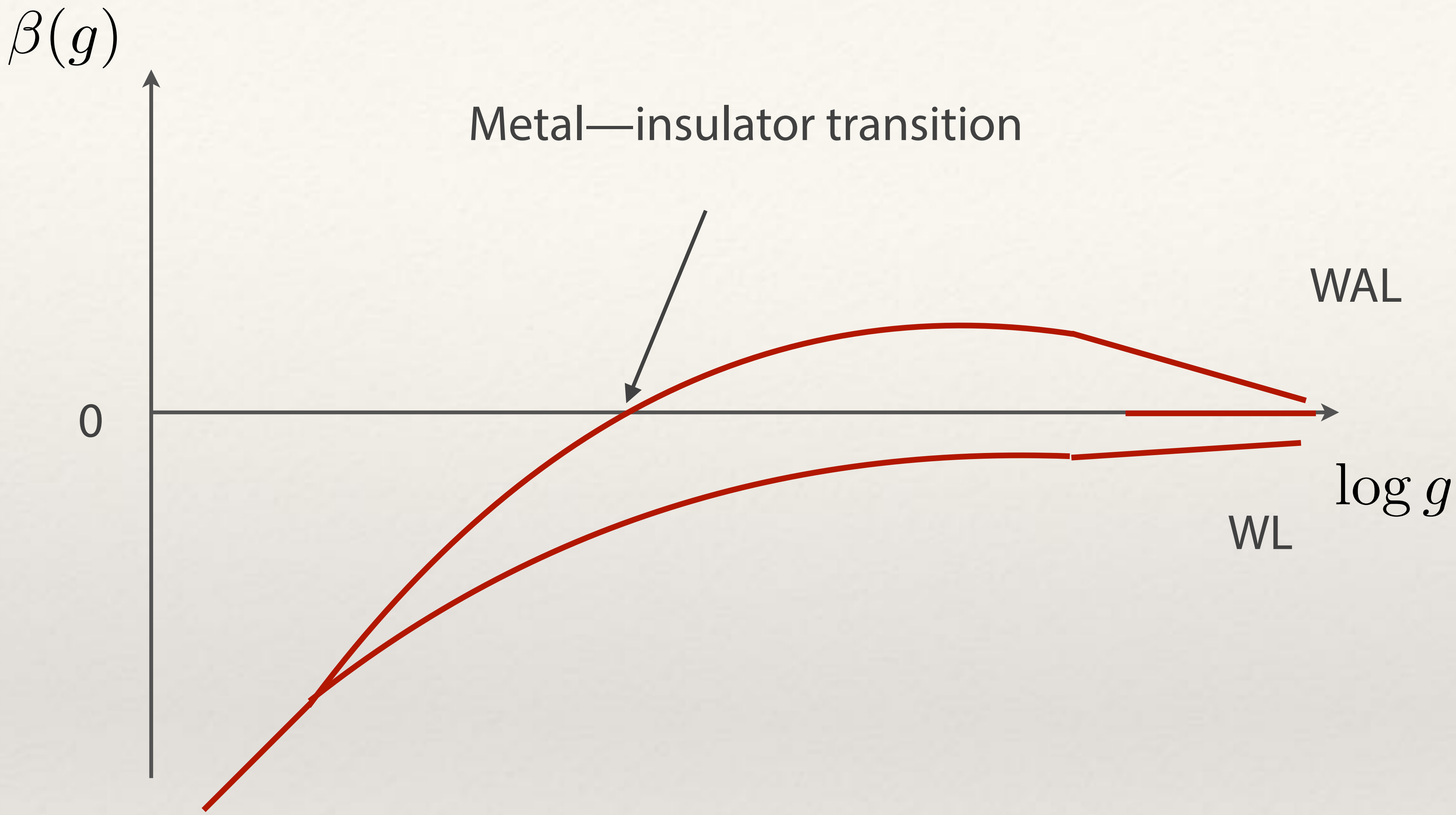
Field theory of diffusion (nonlinear sigma model)







Scaling theory of localization — 2D



Absence of backscattering and perfectly transmitted mode

$$|\psi\rangle = \begin{cases} \sum_n c_{n,\mathrm{L}} \, |n\rangle_{\mathrm{L}} + d_{n,\mathrm{L}} \, |\mathcal{T} n\rangle_{\mathrm{L}} \, , & x \leqslant 0, \\ \sum_n c_{n,\mathrm{R}} \, |n\rangle_{\mathrm{R}} + d_{n,\mathrm{R}} \, |\mathcal{T} n\rangle_{\mathrm{R}} \, , & x \geqslant L, \\ |\Psi\rangle \, , & 0 \leqslant x \leqslant L. \end{cases}$$

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

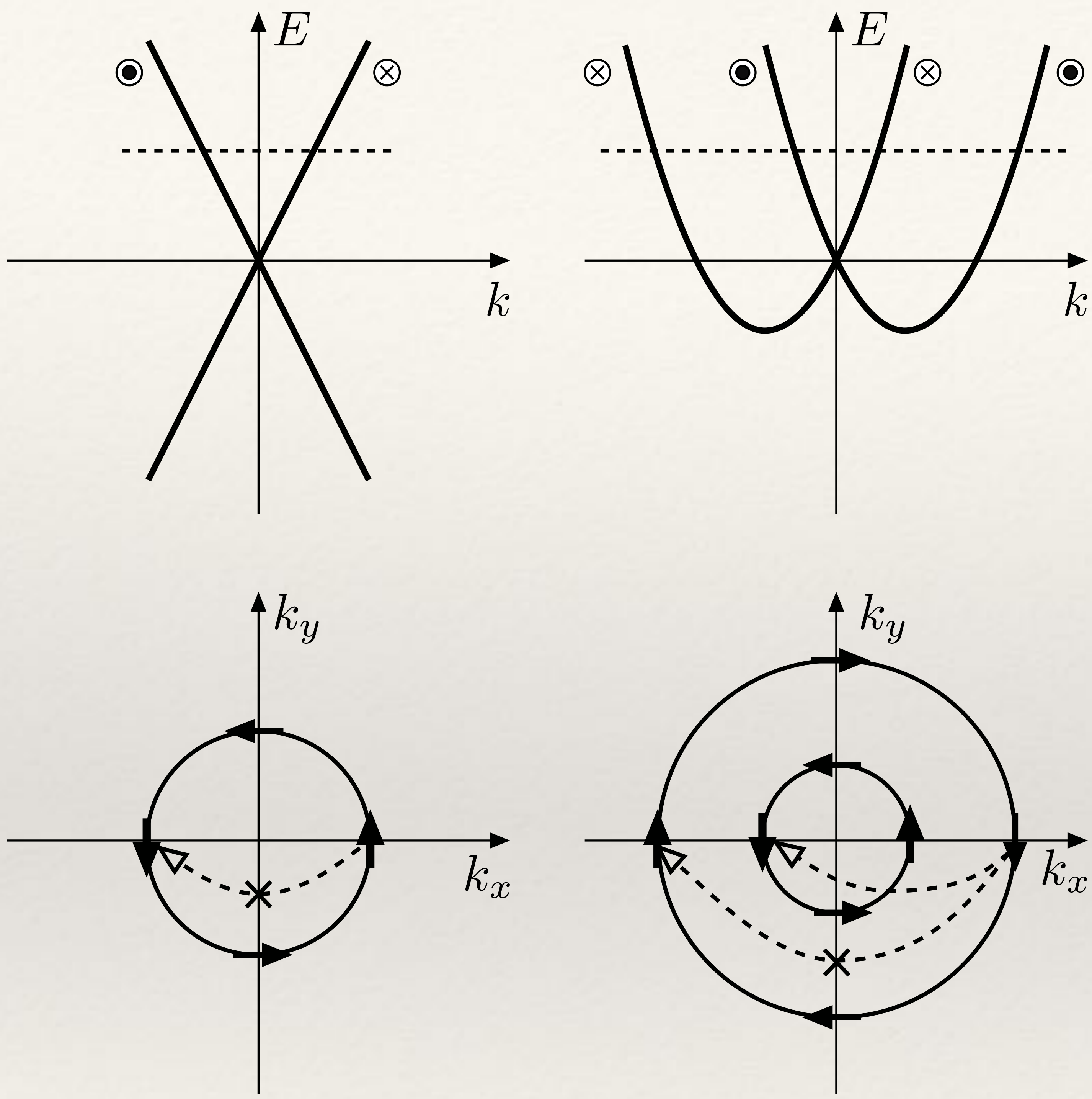
$$\begin{pmatrix} d_{\mathrm{L}} \\ d_{\mathrm{R}} \end{pmatrix} = S \begin{pmatrix} c_{\mathrm{L}} \\ c_{\mathrm{R}} \end{pmatrix}$$

$$|\mathcal{T}\psi\rangle = \begin{cases} \sum_n c^*_{n,\mathrm{L}} \, |\mathcal{T} n\rangle_{\mathrm{L}} - d^*_{n,\mathrm{L}} \, |n\rangle_{\mathrm{L}} \, , & x \leqslant 0, \\ \sum_n c^*_{n,\mathrm{R}} \, |\mathcal{T} n\rangle_{\mathrm{R}} - d^*_{n,\mathrm{R}} \, |n\rangle_{\mathrm{R}} \, , & x \geqslant L, \\ |\mathcal{T}\Psi\rangle \, , & 0 \leqslant x \leqslant L, \end{cases}$$

$$\begin{pmatrix} c^*_{\mathrm{L}} \\ c^*_{\mathrm{R}} \end{pmatrix} = -S \begin{pmatrix} d^*_{\mathrm{L}} \\ d^*_{\mathrm{R}} \end{pmatrix}$$

$$S^T = -S$$

Absence of backscattering and perfectly transmitted mode



$$r = -r^T$$

$$\Rightarrow \quad r_{nn} = 0$$

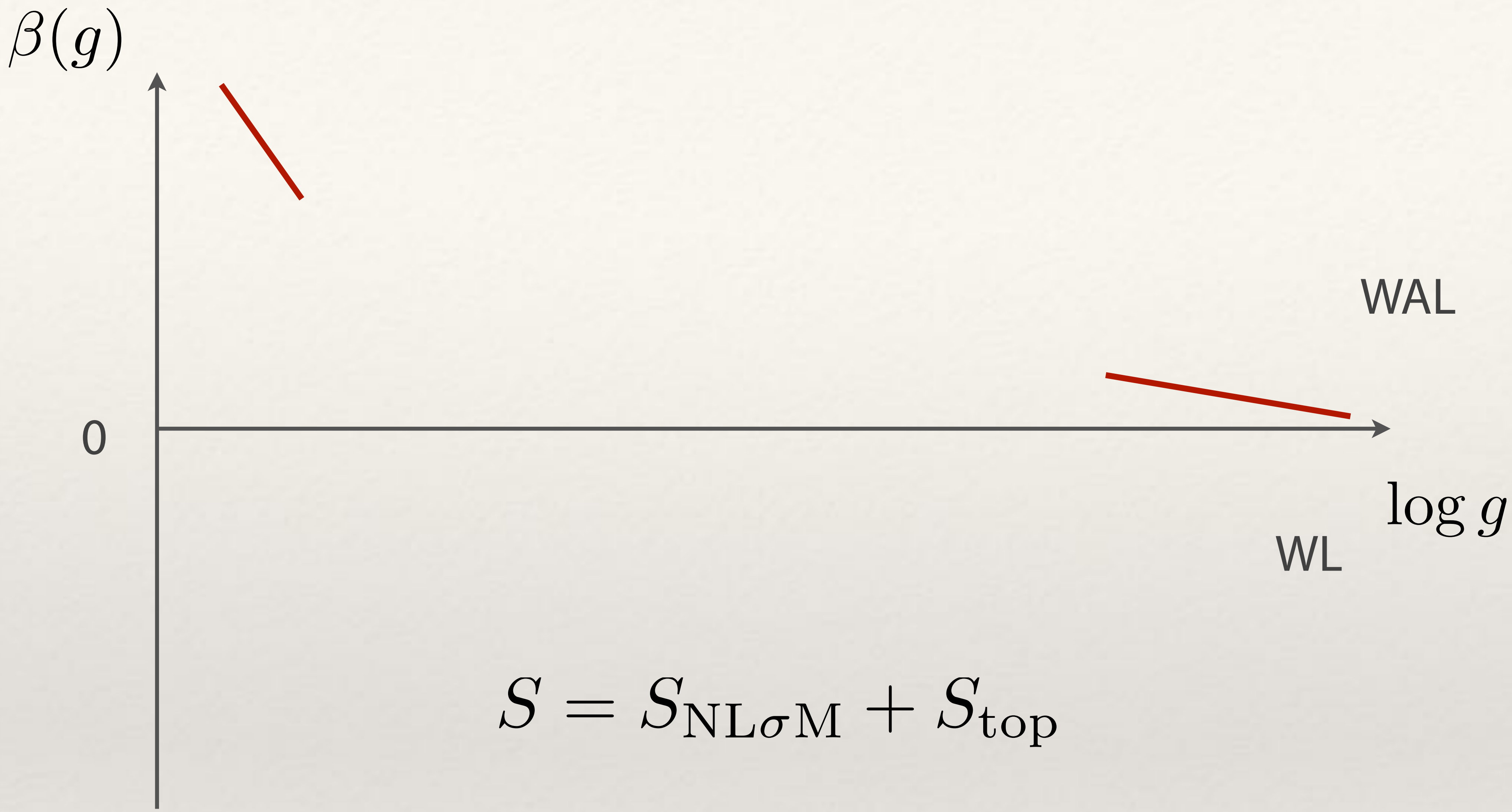
$$\det r = \det -r^T$$

$$= (-1)^N \det r$$

if N odd $\Rightarrow$

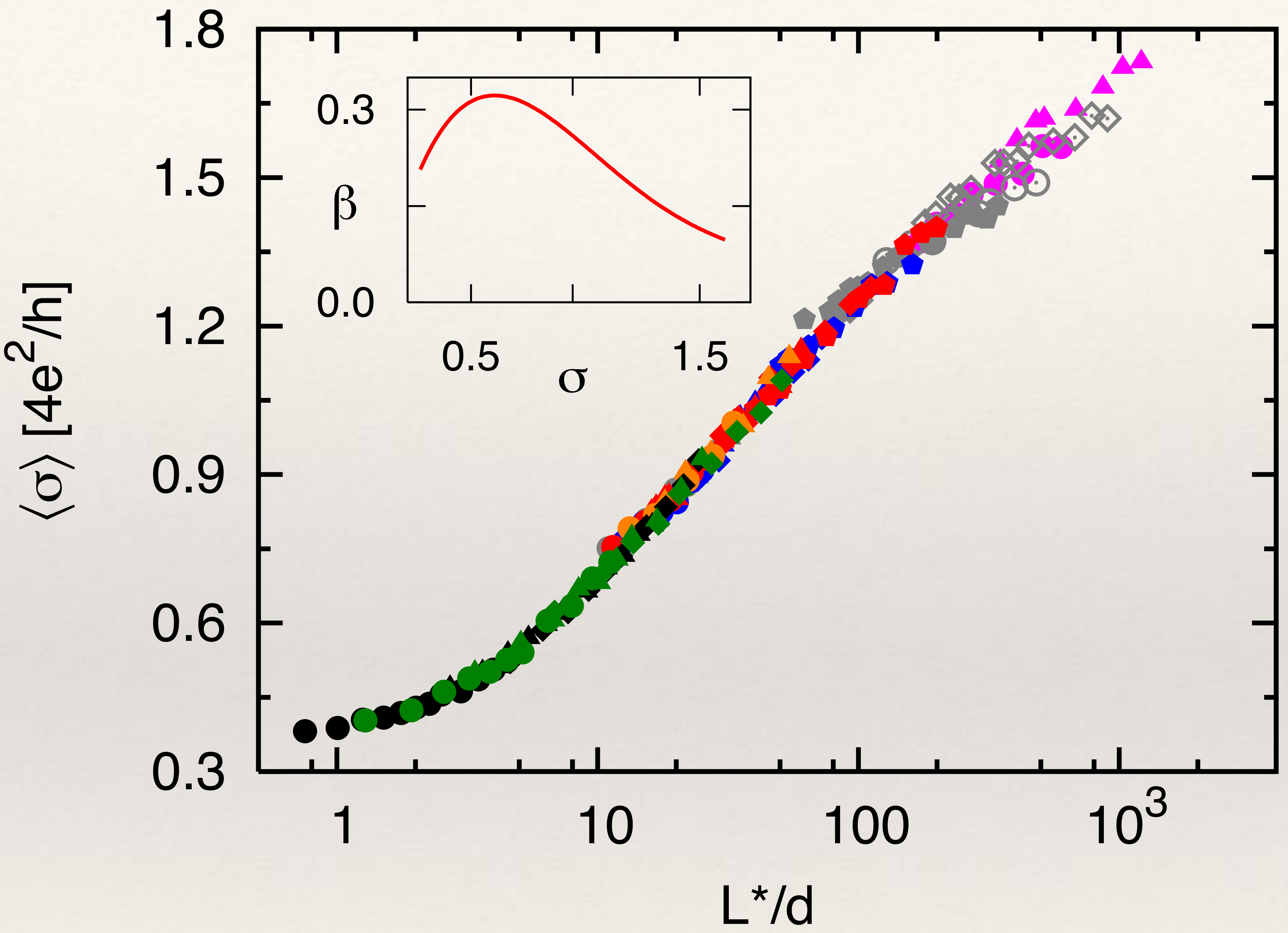
$$\det r = 0$$

Scaling theory of localization



$$S = S_{\mathrm{NL}\sigma\mathrm{M}} + S_{\mathrm{top}}$$

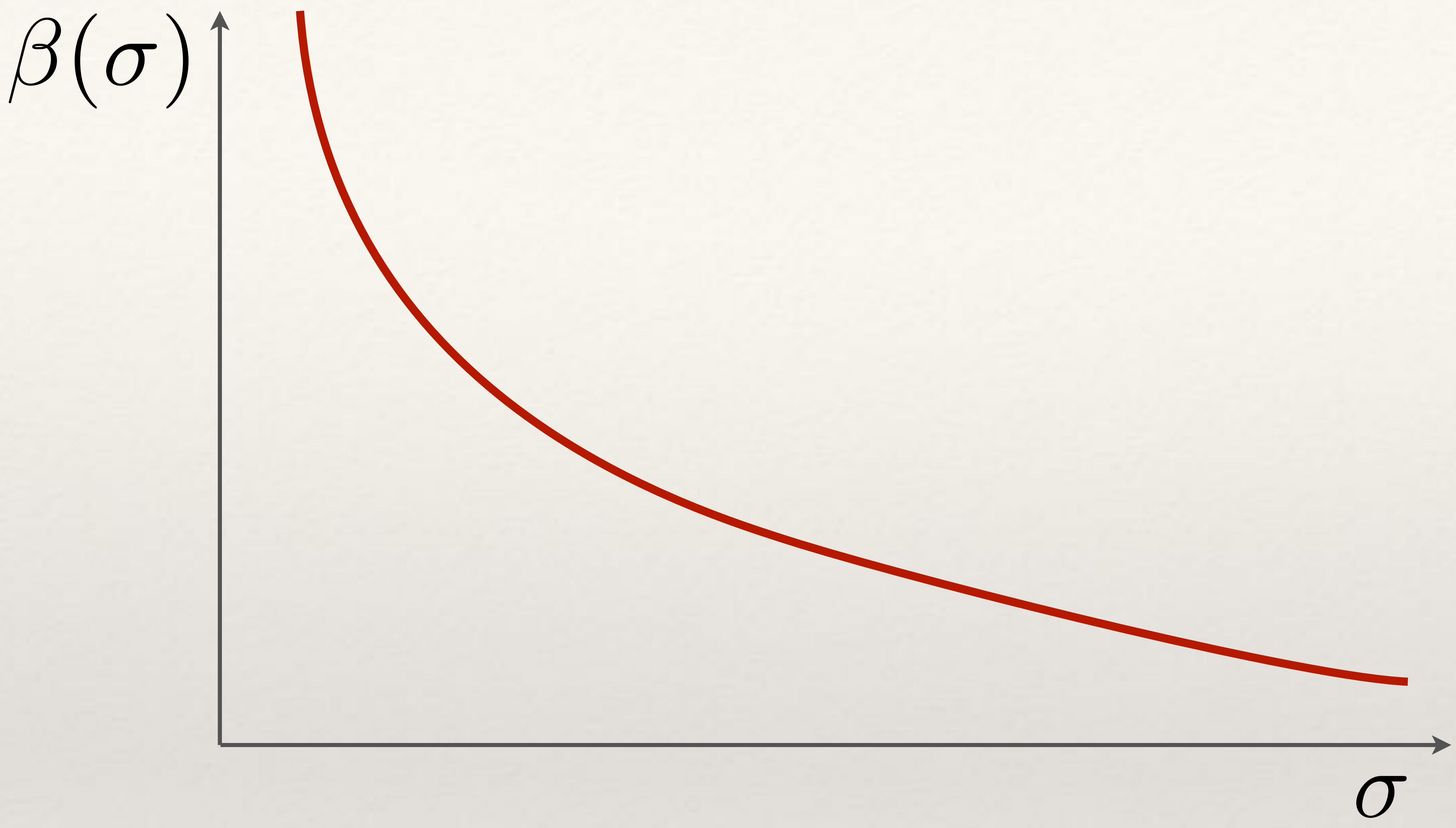
Absence of localisation



JHB, Tworzydlo, Brouwer, Beenakker, PRL 2007

Nomura, Koshino, Ryu PRL 2007

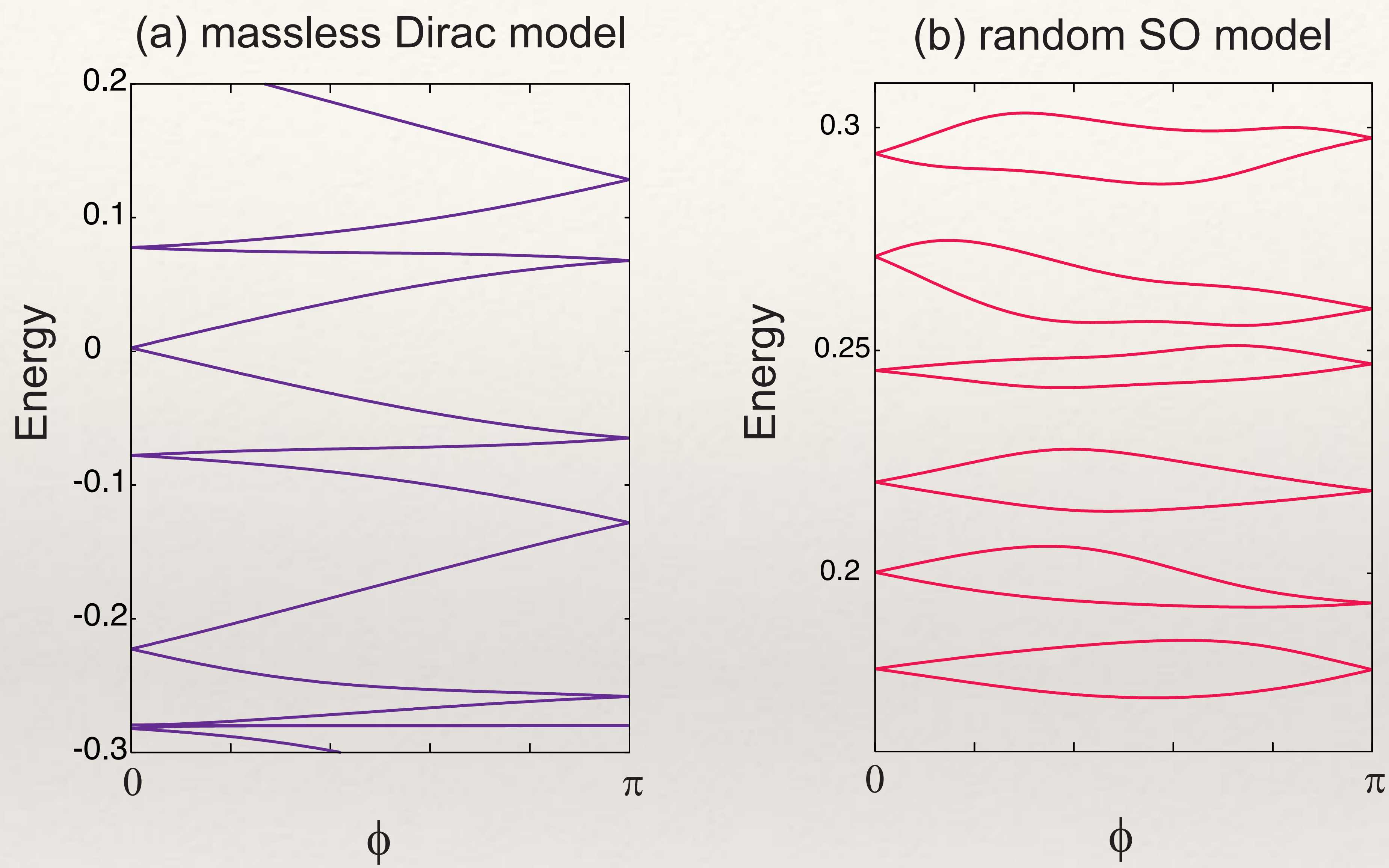
Surface of a topological insulator can not be localized



Note: stronger condition than being gapless

JHB, Tworzydlo, Brouwer, Beenakker, PRL 2007
Nomura, Koshino, Ryu PRL 2007

Topology of energy bands

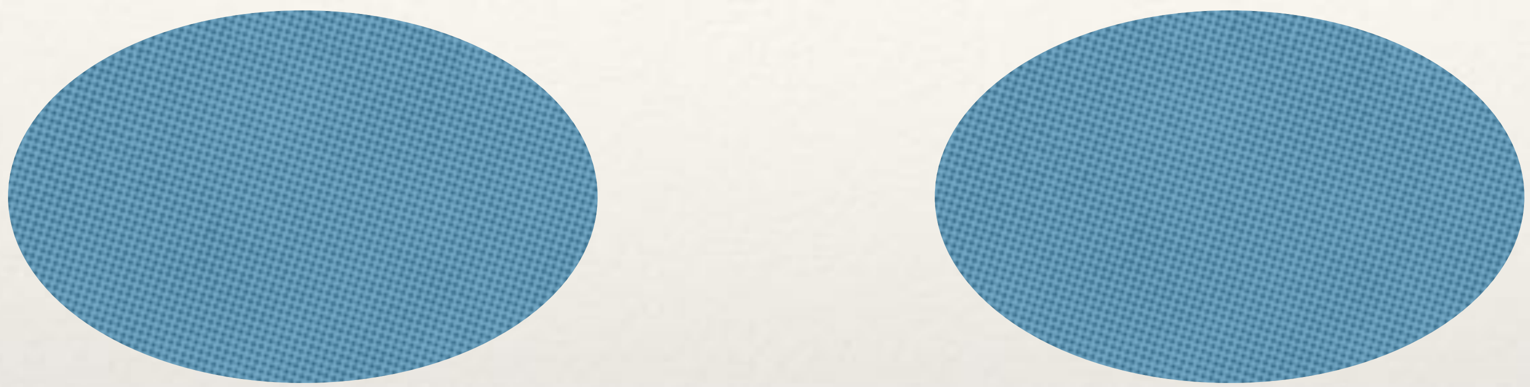


Nomura, Koshino, Ryu PRL 2007

Symmetric spaces and topological terms

$$S = S[Q] \qquad\qquad Q \in O(2n)/O(n) \times O(n)$$

Symmetric space with two disconnected components



Topological term gives a different sign in the action to different components

A topological insulator in d dimensions has a d-1 dimensional surface that can not be localized. The non-linear sigma model describing the surface correspondingly has a topological term

The periodic table of topological insulators

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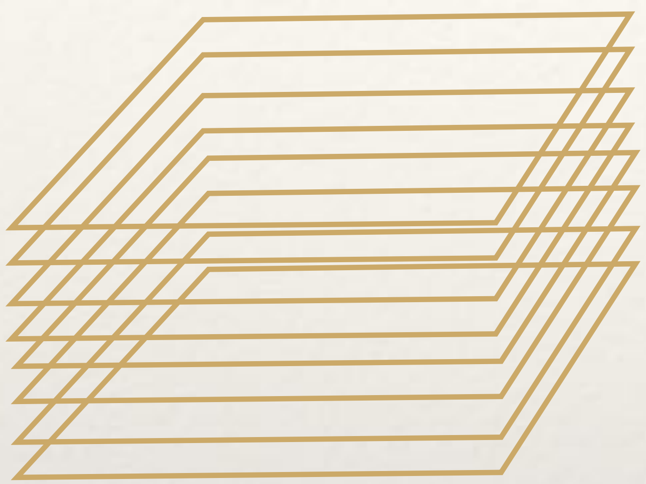
AI	+1	0	0	U(N)/O(N)	Sp(2n)/ Sp(n)xSp(n)	Orthogonal	$\mathbb{Z}$	0	0	0
BDI	+1	+1	1	O(N+M)/ O(N)xO(M)	U(2n)/Sp(2n)	Chiral orthogonal	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
D	0	+1	0	SO(2N)	O(2n)/U(n)	BdG	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	+1	1	SO(2N)/U(N)	O(2n)	BdG	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	U(2N)/Sp(2N)	O(2n)/ O(n)xO(n)	Symplectic	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	Sp(N+M)/ Sp(N)xSp(M)	U(2n)/O(2n)	Chiral symplectic	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	BdG	0	0	$2\mathbb{Z}$	0
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	BdG	0	0	0	$2\mathbb{Z}$

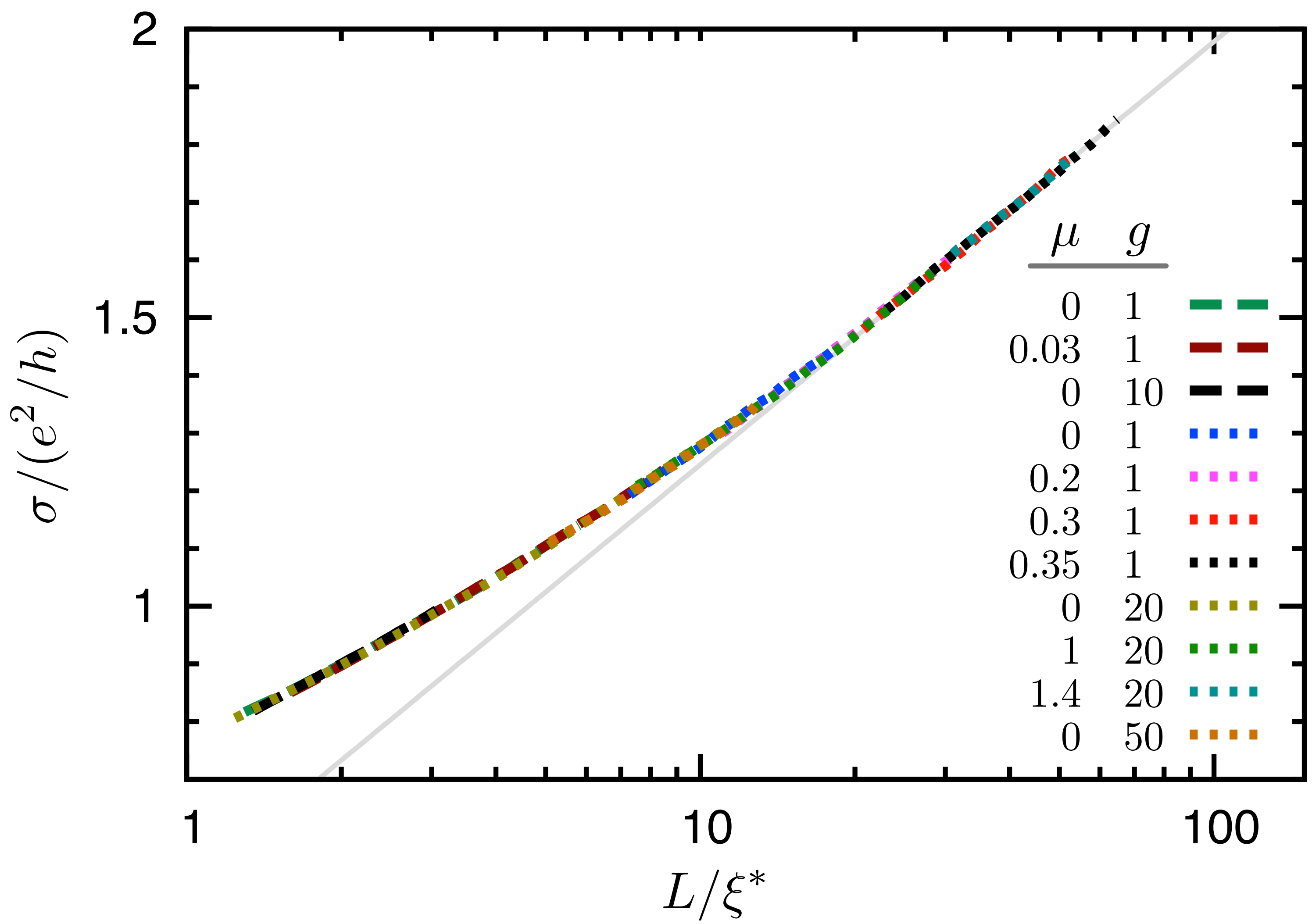
Ryu, Schneider, Furusaki, Ludwig NJP (2010)

Weak topological insulators

$(\nu_0, \boldsymbol{\nu})$ Strong and weak indices

Can think of as stacking of lower dimensional strong topological insulators

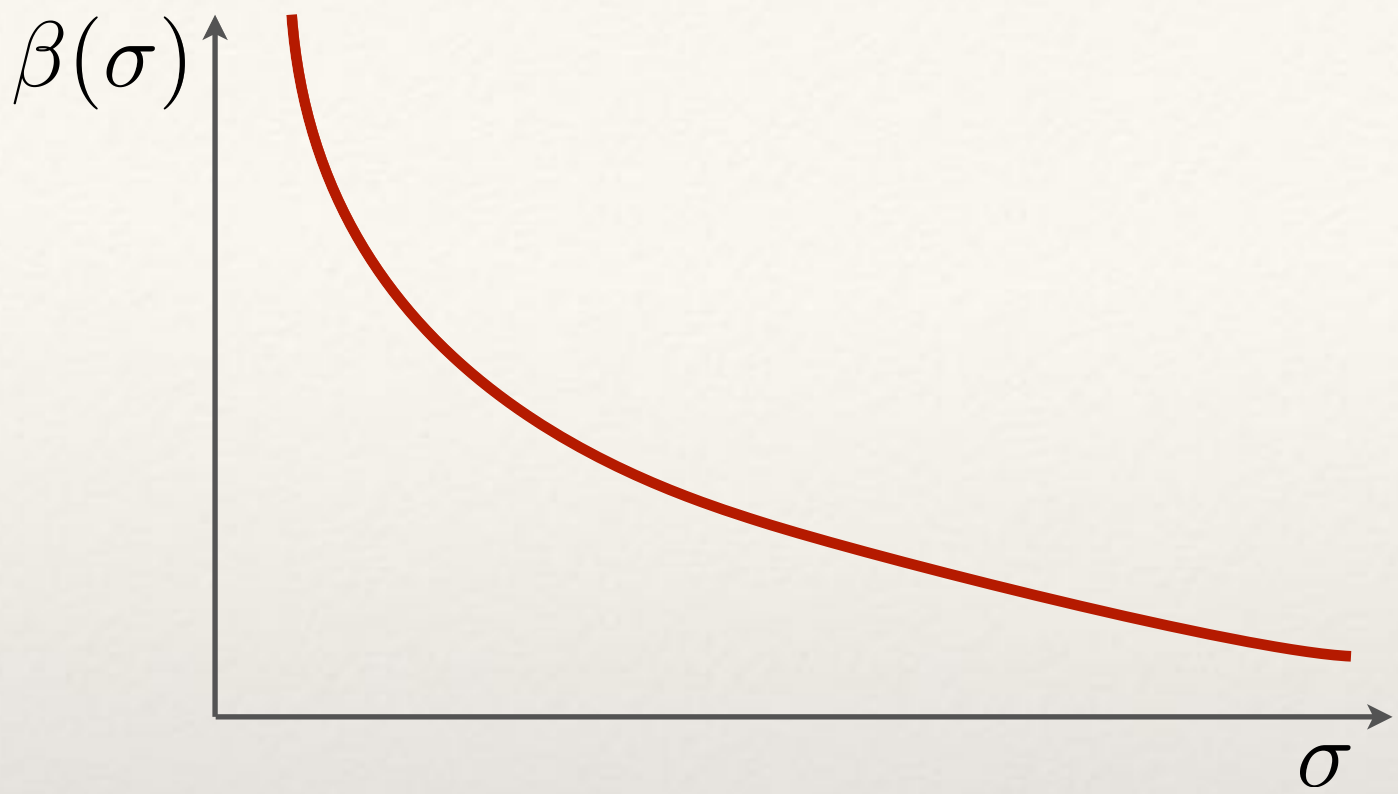




Does not localize unless average translation symmetry broken

Mong, JHB, Moore PRL 2012

Summary of lecture 1



Topological insulators characterized by a surface that doesn't localize

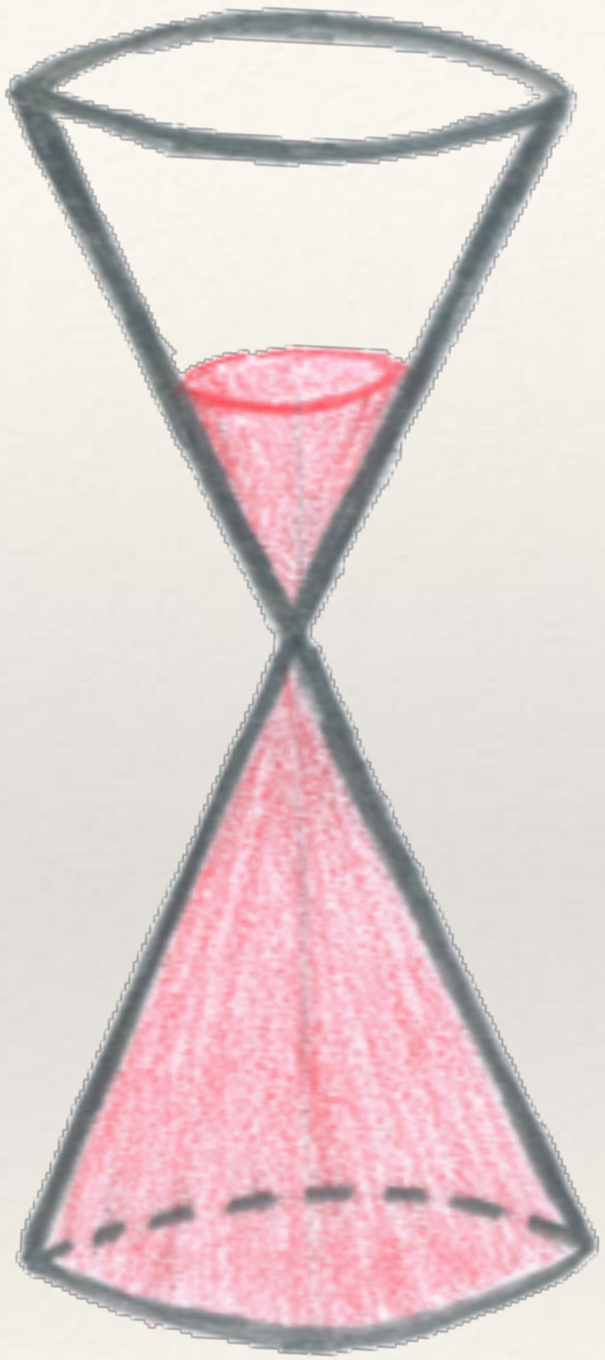
In 3D top. ins. disorder drives the surface into a symplectic metal, characterized by weak anti-localization

Topological insulator nanowires

Jens Hjörleifur Bárðarson

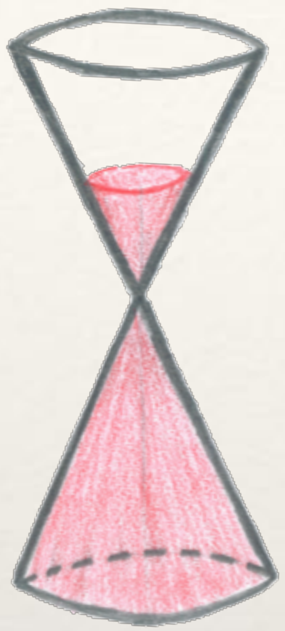
Max Planck Institute PKS, Dresden

KTH Royal Institute of Technology, Stockholm



Review: **JHB** and J. E. Moore, Rep. Prog. Phys. 2013

Surface of a 3D topological insulator is a supermetal



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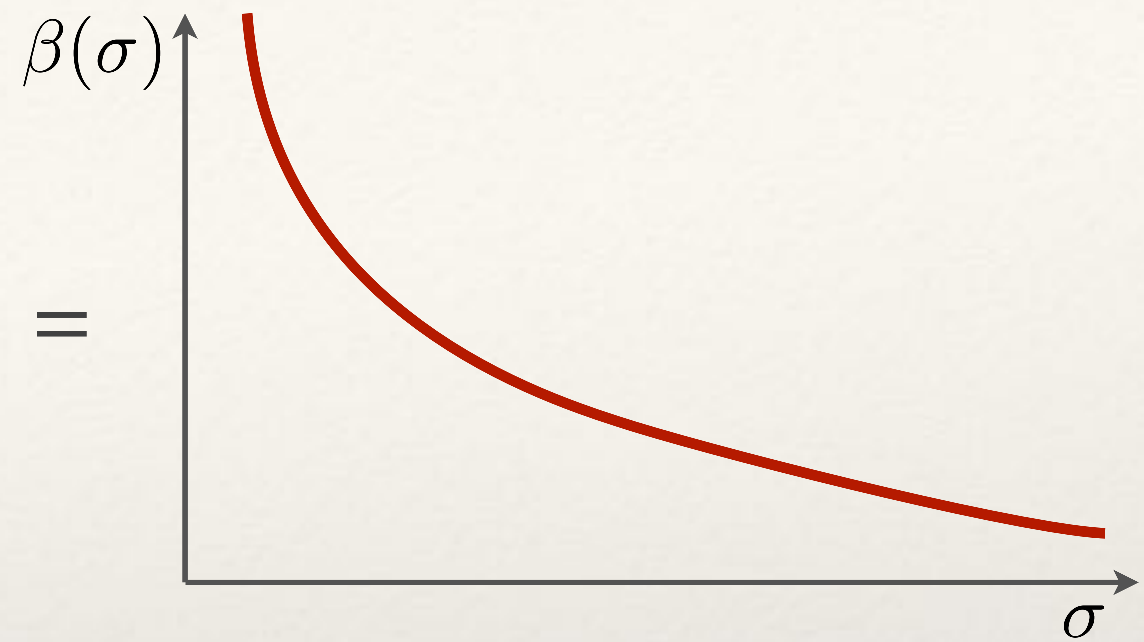
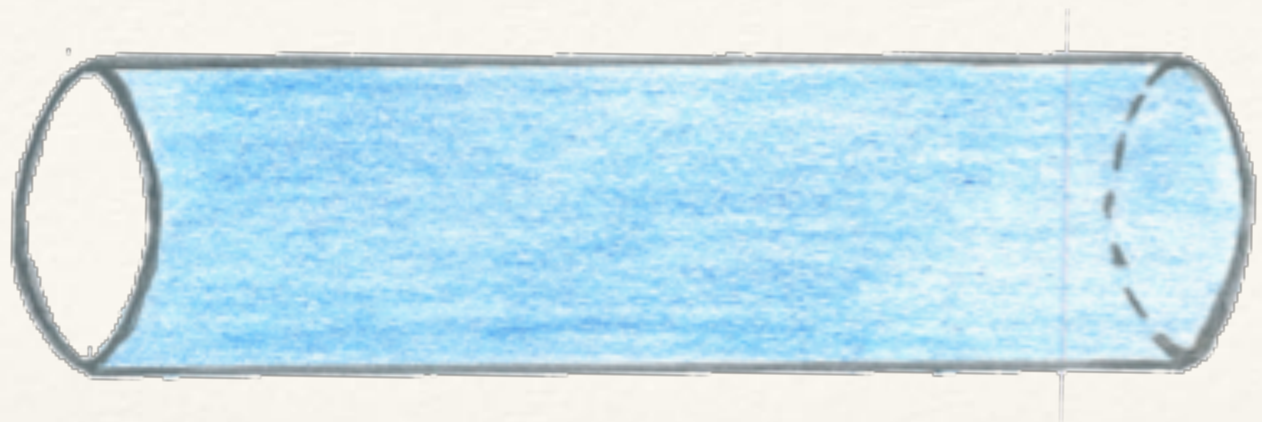


$$H = p_x \sigma_x + p_y \sigma_y + V(\mathbf{r})$$

JHB, Tworzydlo, Brouwer, Beenakker, PRL 2007

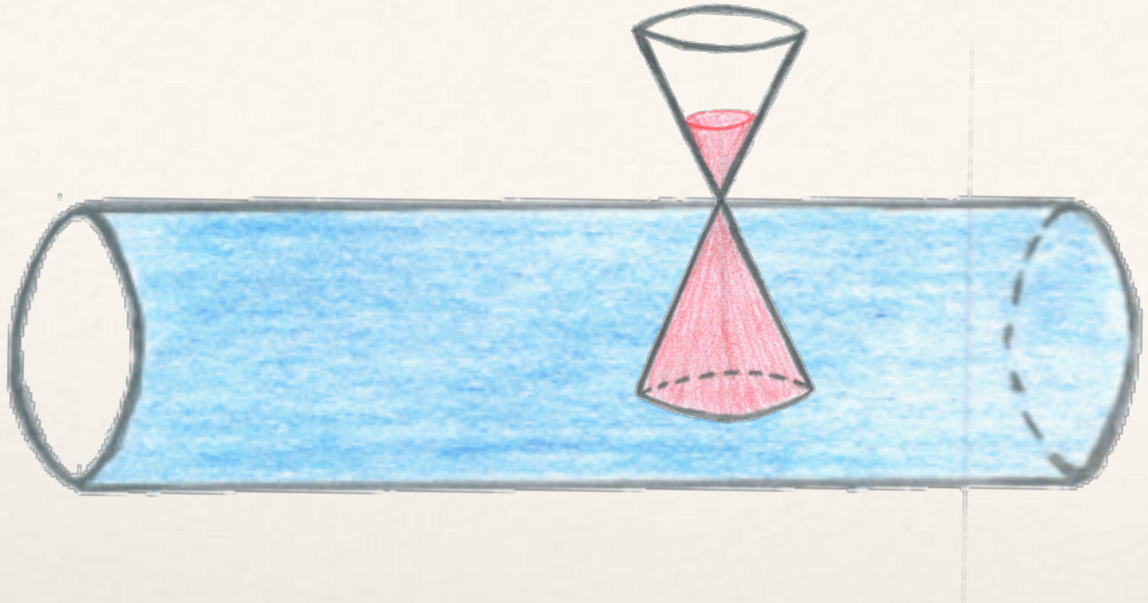
Nomura, Koshino, Ryu PRL 2007

Topological insulator nanowires host various interesting modes



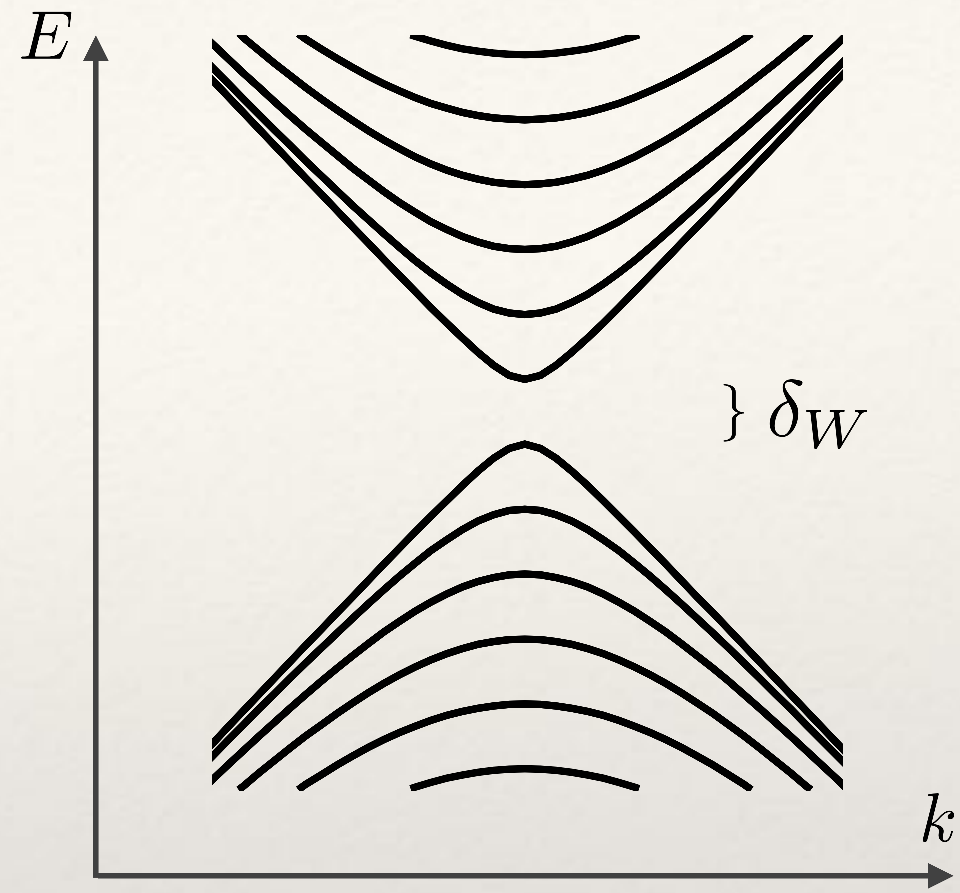
Perfectly transmitted mode —— Chiral modes —— Majorana modes

Dirac fermion on a curved surface



$$H = p_x \sigma_x + p_y \sigma_y$$

$$\psi(\theta + 2\pi) = -\psi(\theta)$$

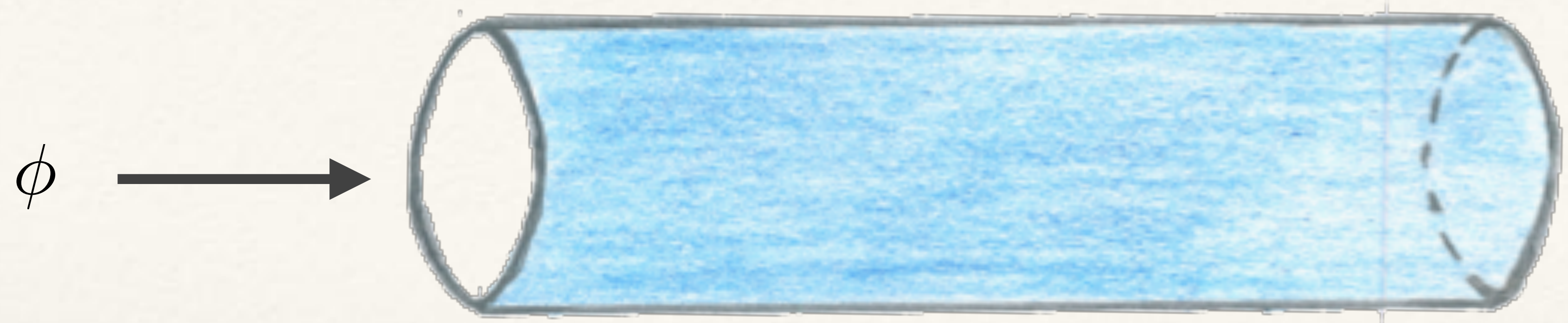


$$\delta_W = \frac{\hbar v_F 2\pi}{W} \underset{@380\ \mathrm{nm}}{=} 6\ \mathrm{meV}$$

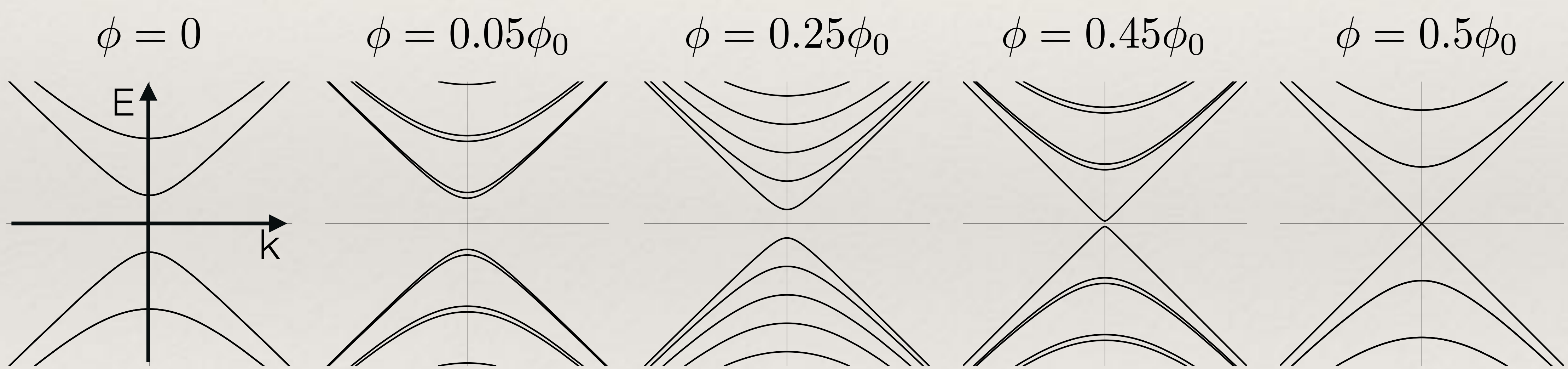
JHB, Brouwer, Moore PRL 2010

Ran, Vishwanath, Lee PRL 2008; Rosenberg, Guo, Franz PRB(R) 2010; Ostrovsky, Gornyi, Mirlin PRL 2010

Aharonov-Bohm flux modifies spectrum

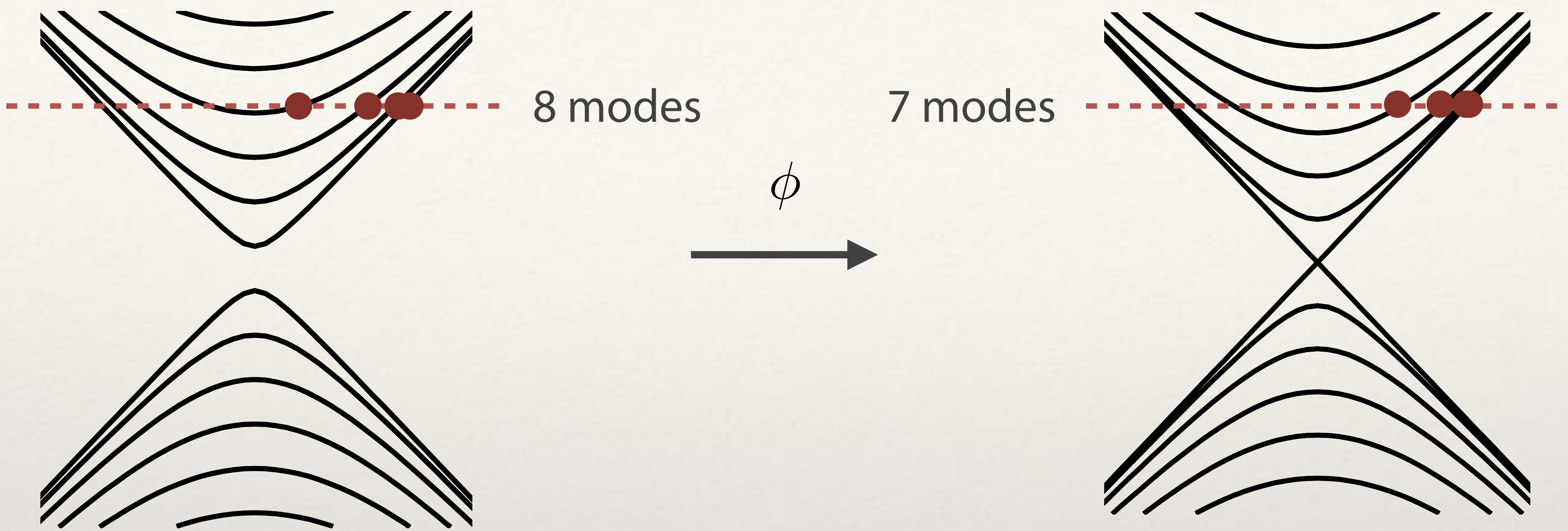


$$\psi(\theta + 2\pi) = e^{i\pi + 2i\pi\phi/\phi_0}\psi(\theta)$$



JHB, Brouwer, Moore PRL 2010

From even to odd — the perfectly transmitted mode emerges



$$THT^{-1} = H \qquad T^2 = -1$$

$\Rightarrow$

$$S^T = -S \qquad r^T = -r$$

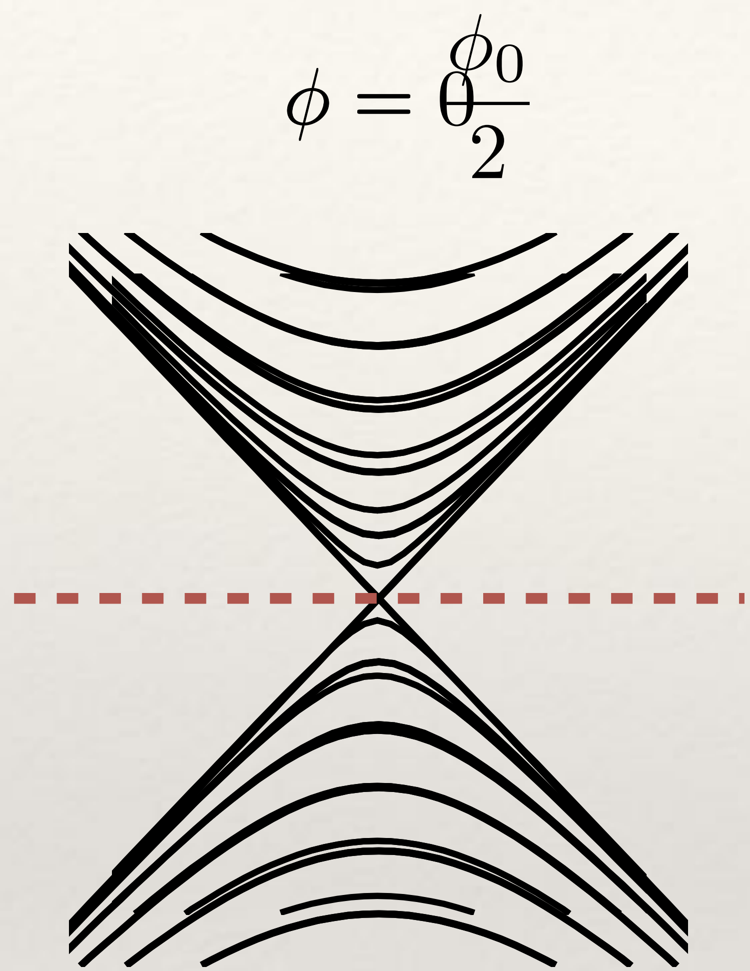
$\Rightarrow$

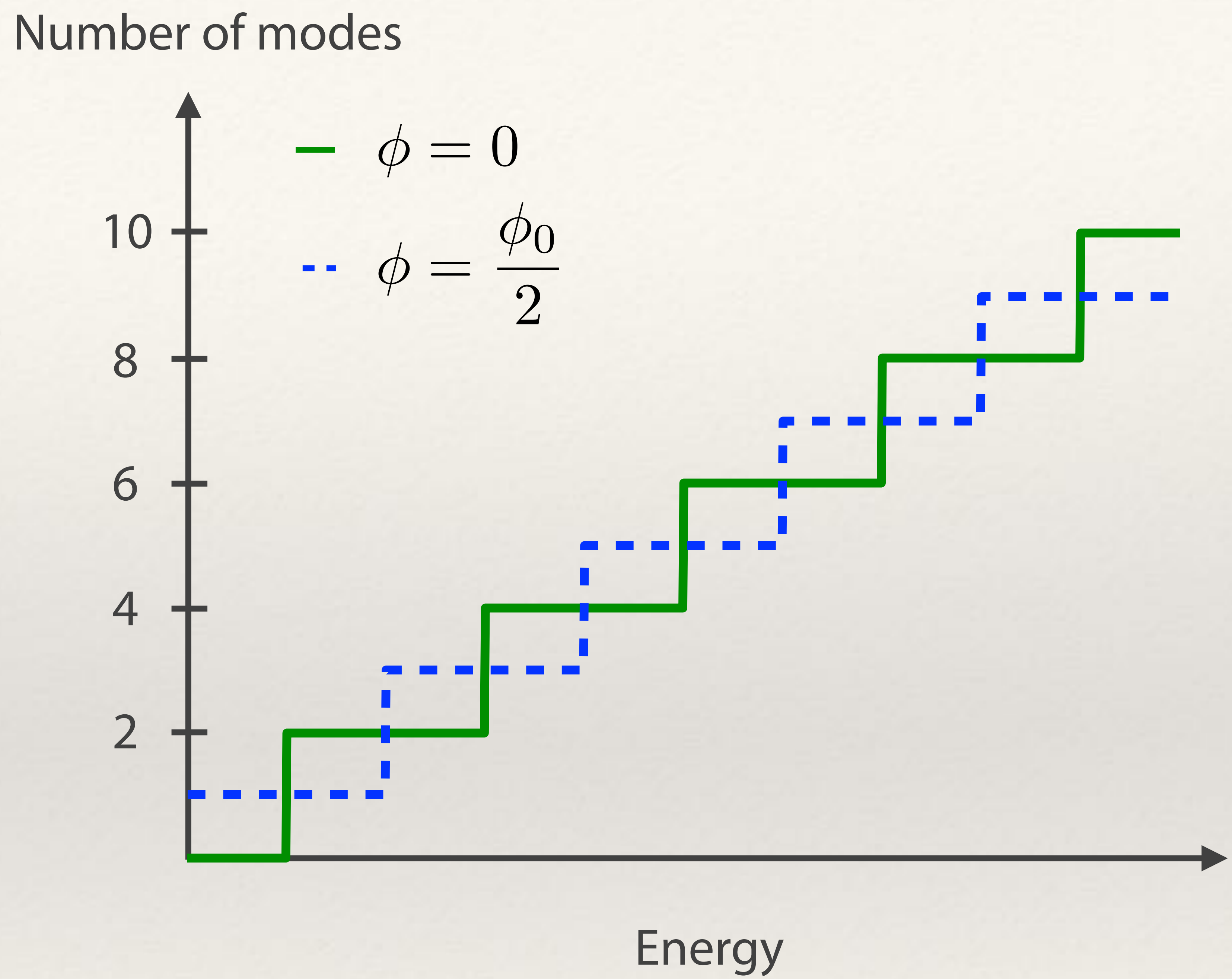
$$\det r = (-1)^N \det r$$

$$G \geq \frac{e^2}{h} \qquad N \text{ odd}$$

Ando, Suzuura J. Phys. Soc. Jpn. 2002;
JHB, Brouwer, Moore PRL 2010; **JHB** J. Phys. A: Math. Theor. 2008

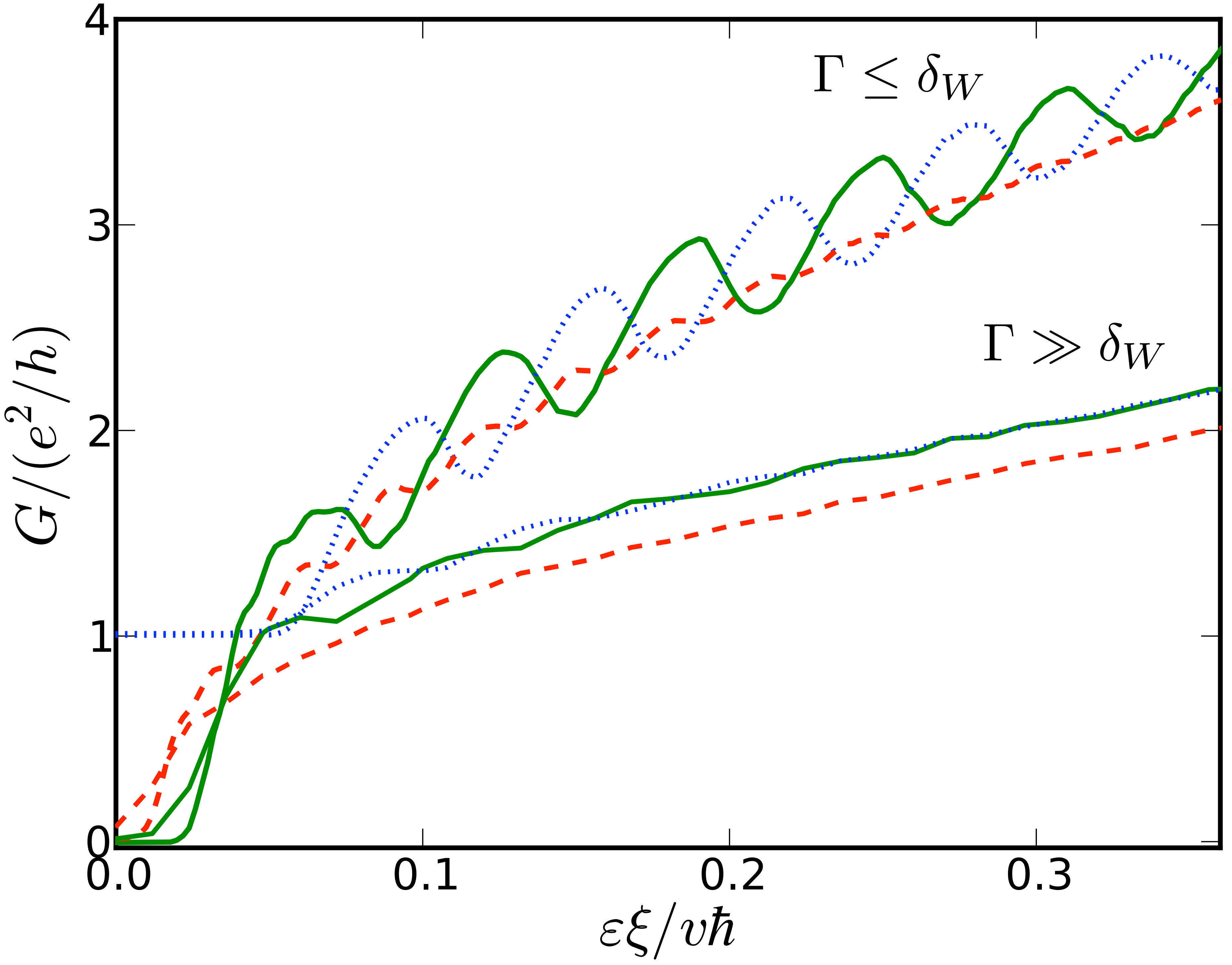
Number of modes depends on energy and flux





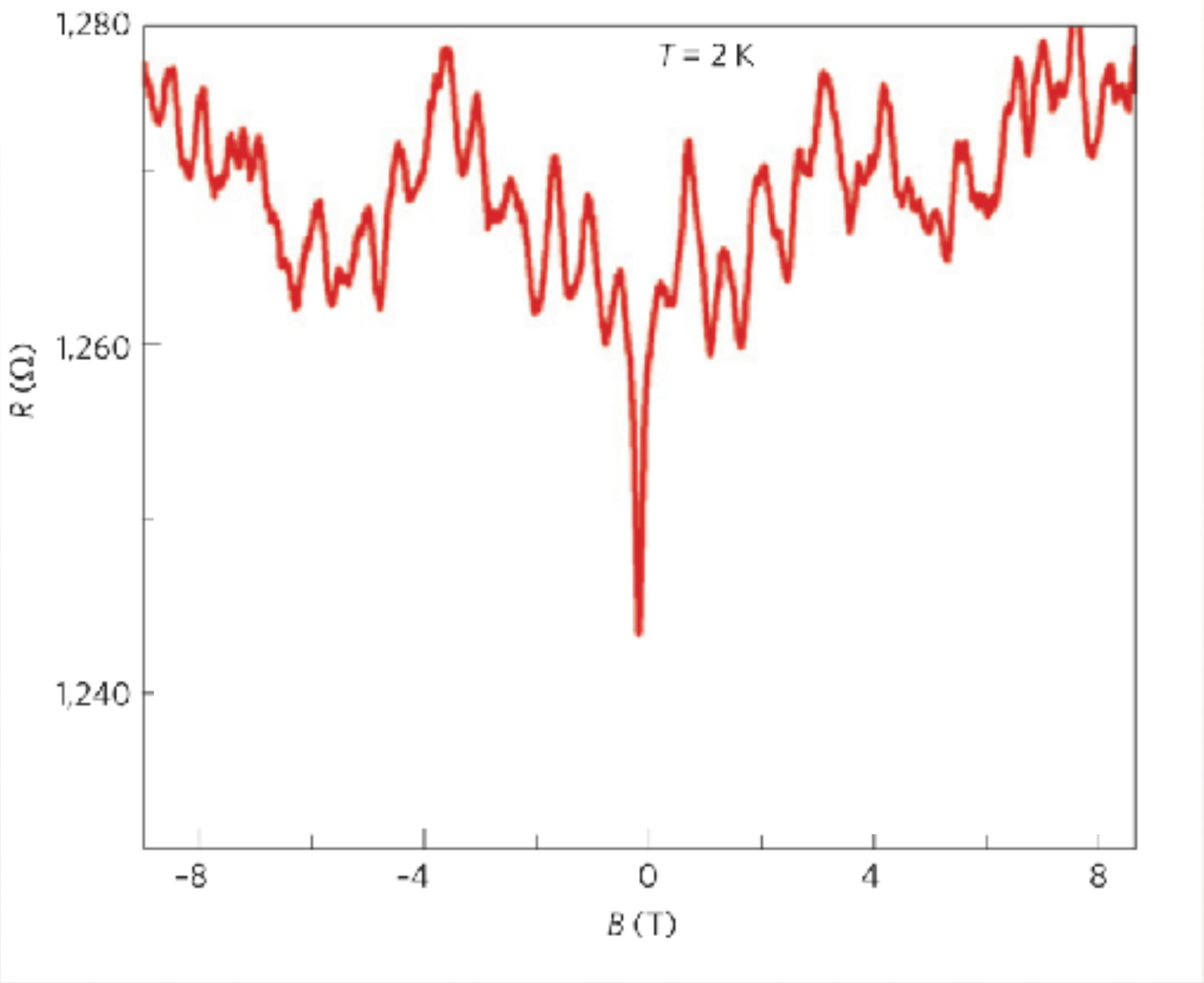
JHB, Brouwer, Moore PRL 2010

Conductance vs. chemical potential

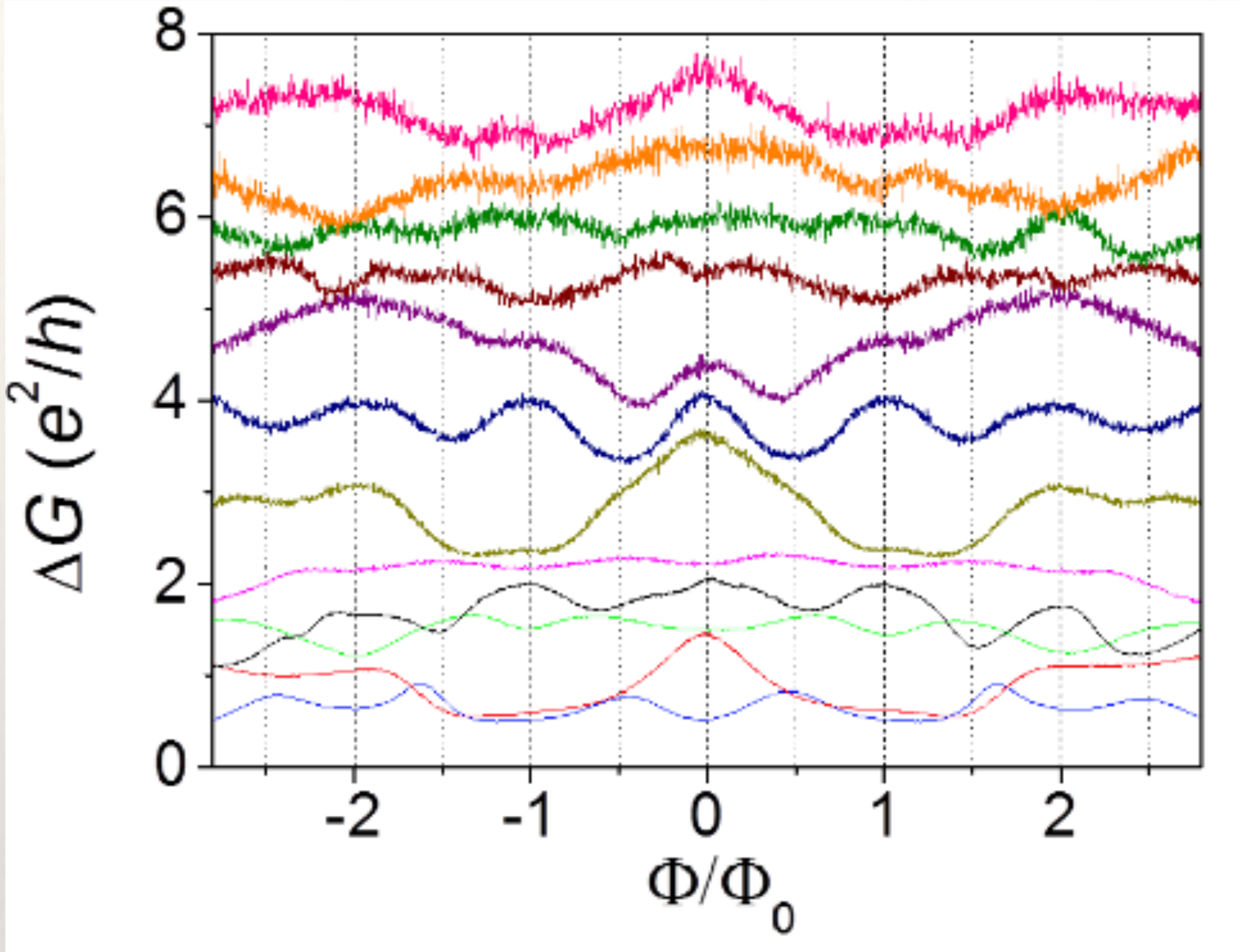


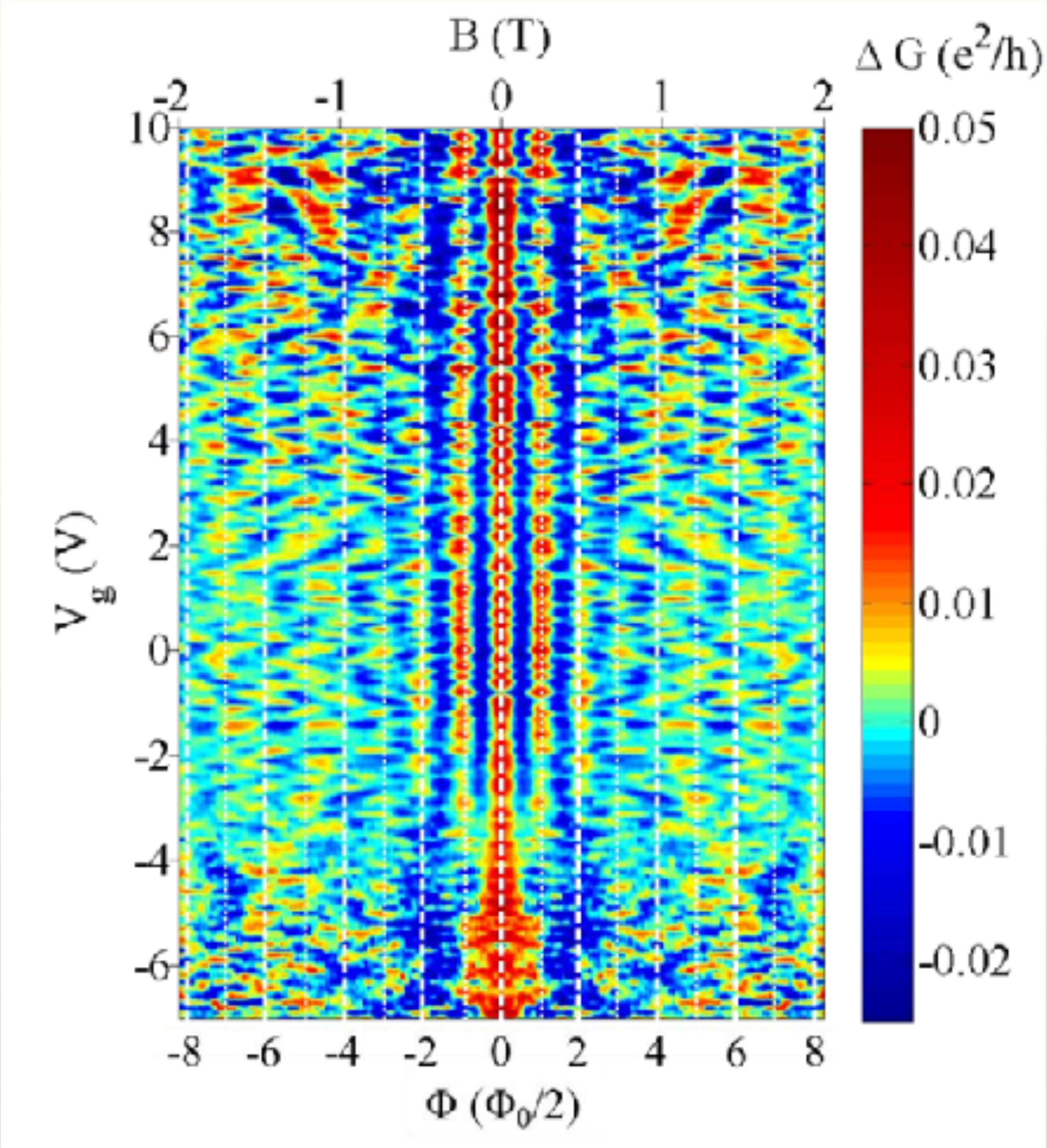
JHB, Brouwer, Moore PRL 2010

Experimental status



Peng, …, Chui, Nature Mater. 2010

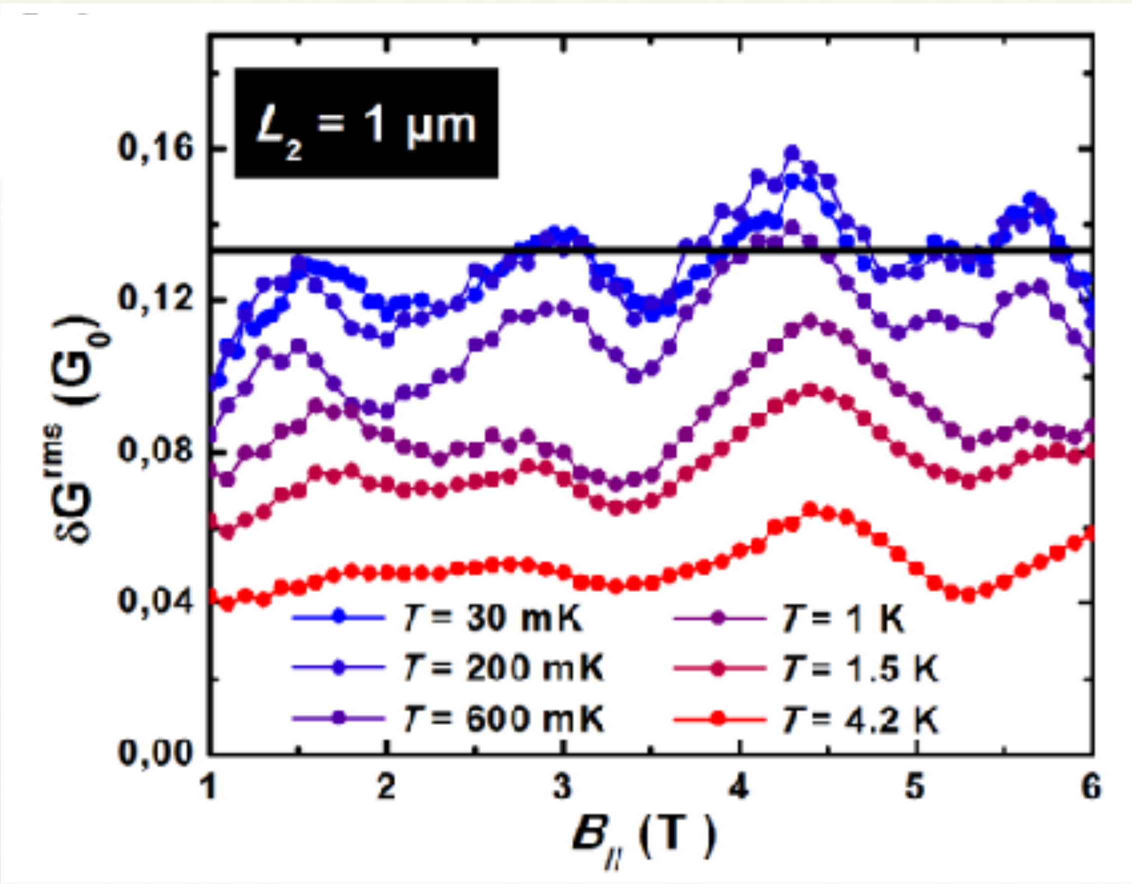


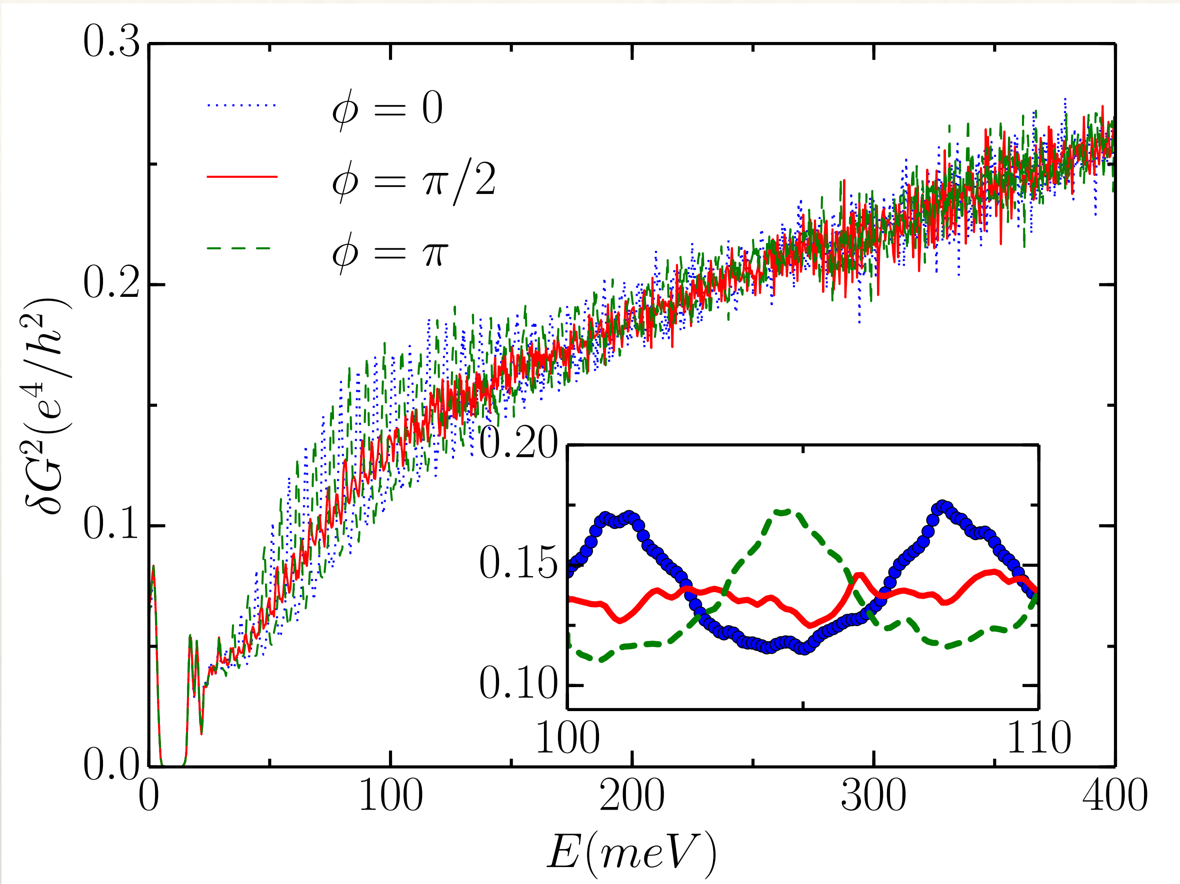


S. Cho,…,N. Mason, Nature Comm. 2015

L. Jauregui,…,Y.P. Chen, Nature Nanotechnol. 2016

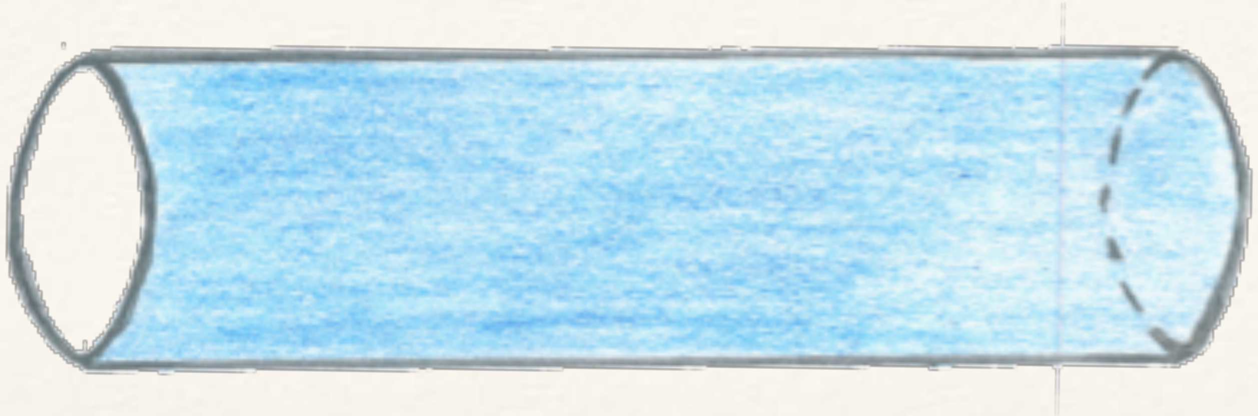
Theory



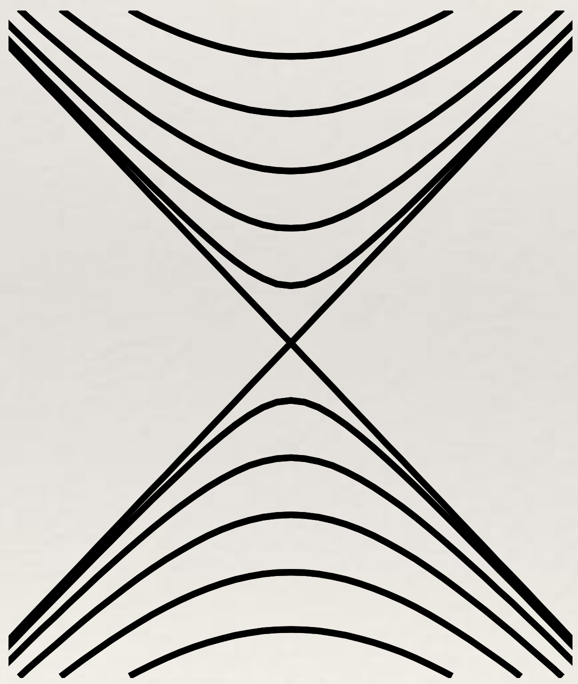


Dufouleur, Veyrat, Xypakis, **JHB**,…, Giraud Sci. Rep. 2017

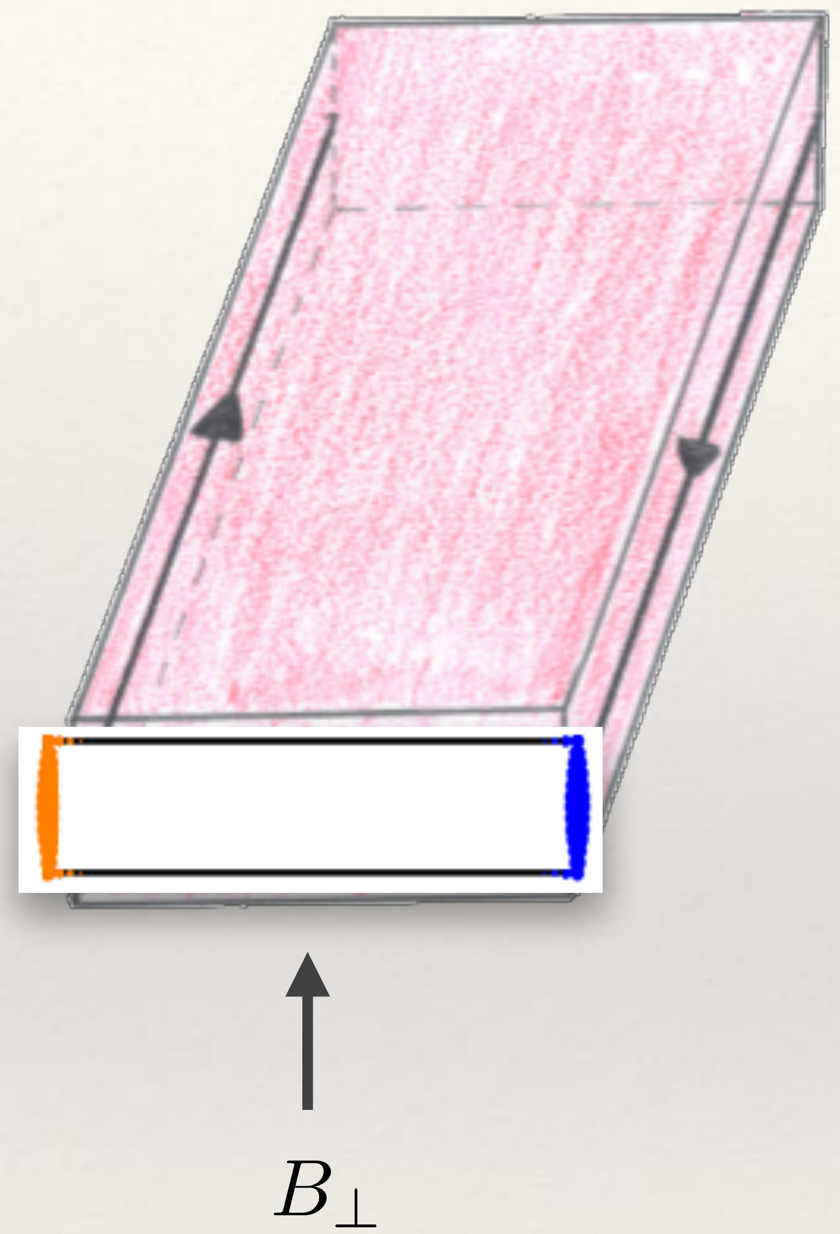
Topological insulator nanowires host various interesting modes

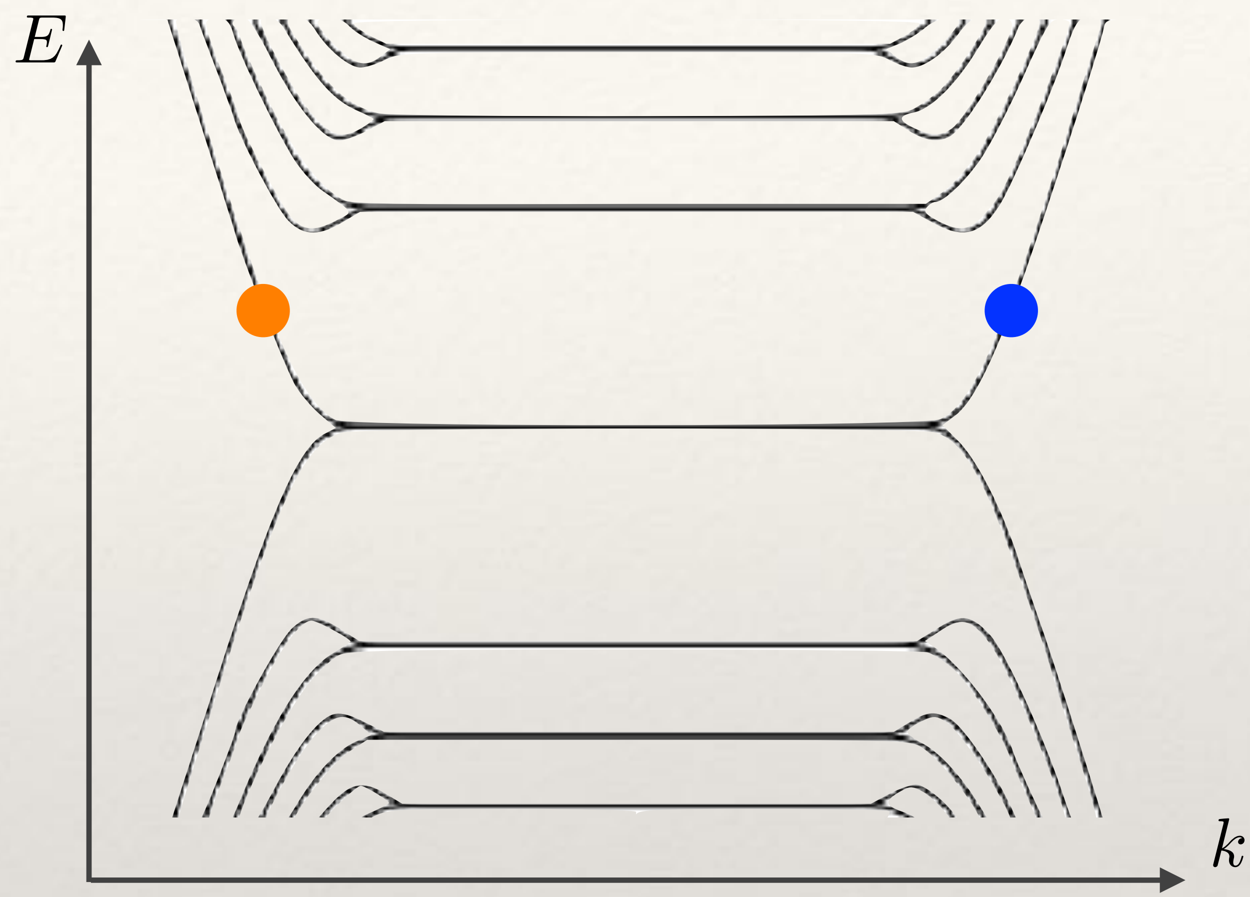


Perfectly transmitted mode —— Chiral modes —— Majorana modes



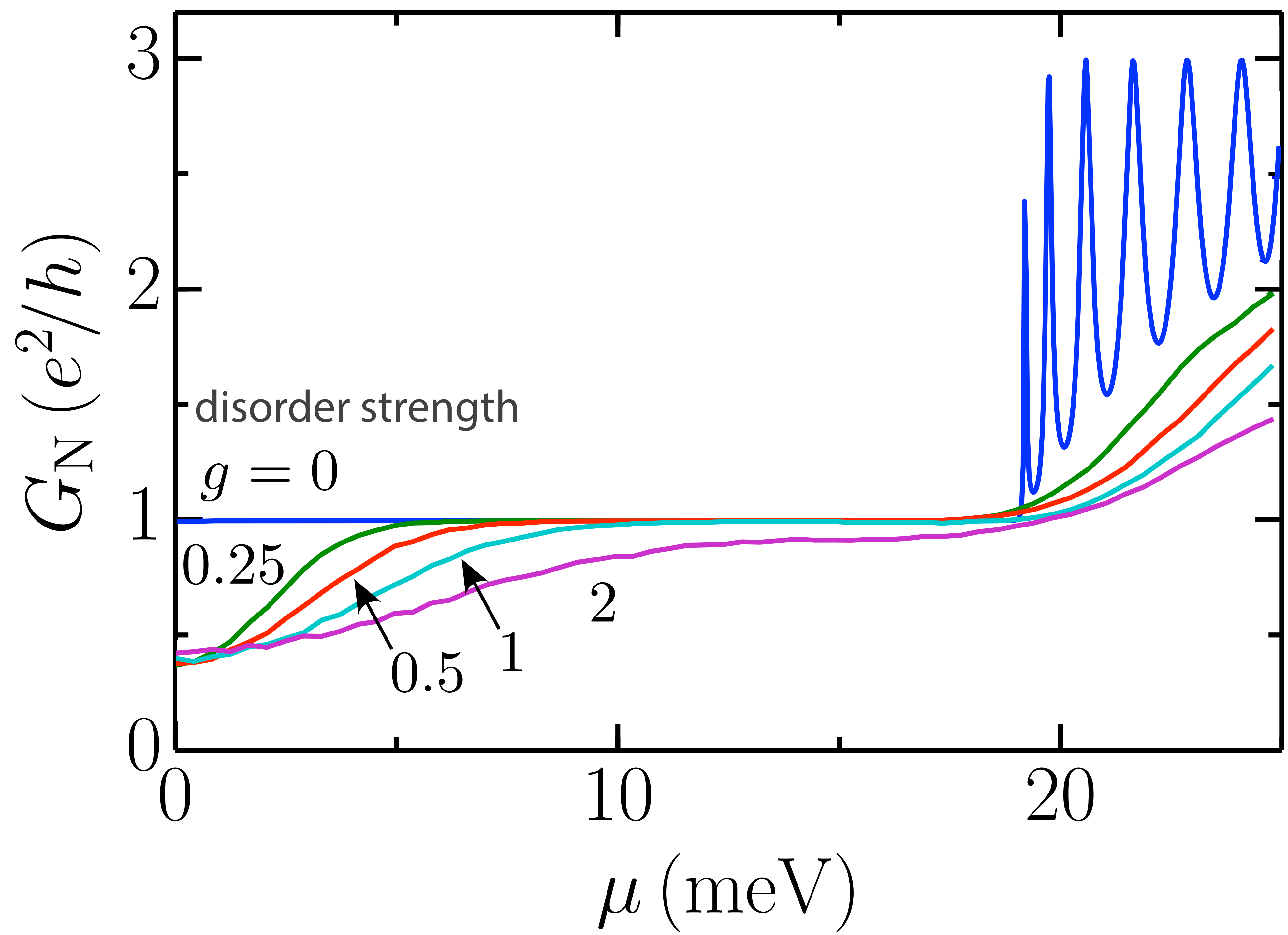
Going chiral…





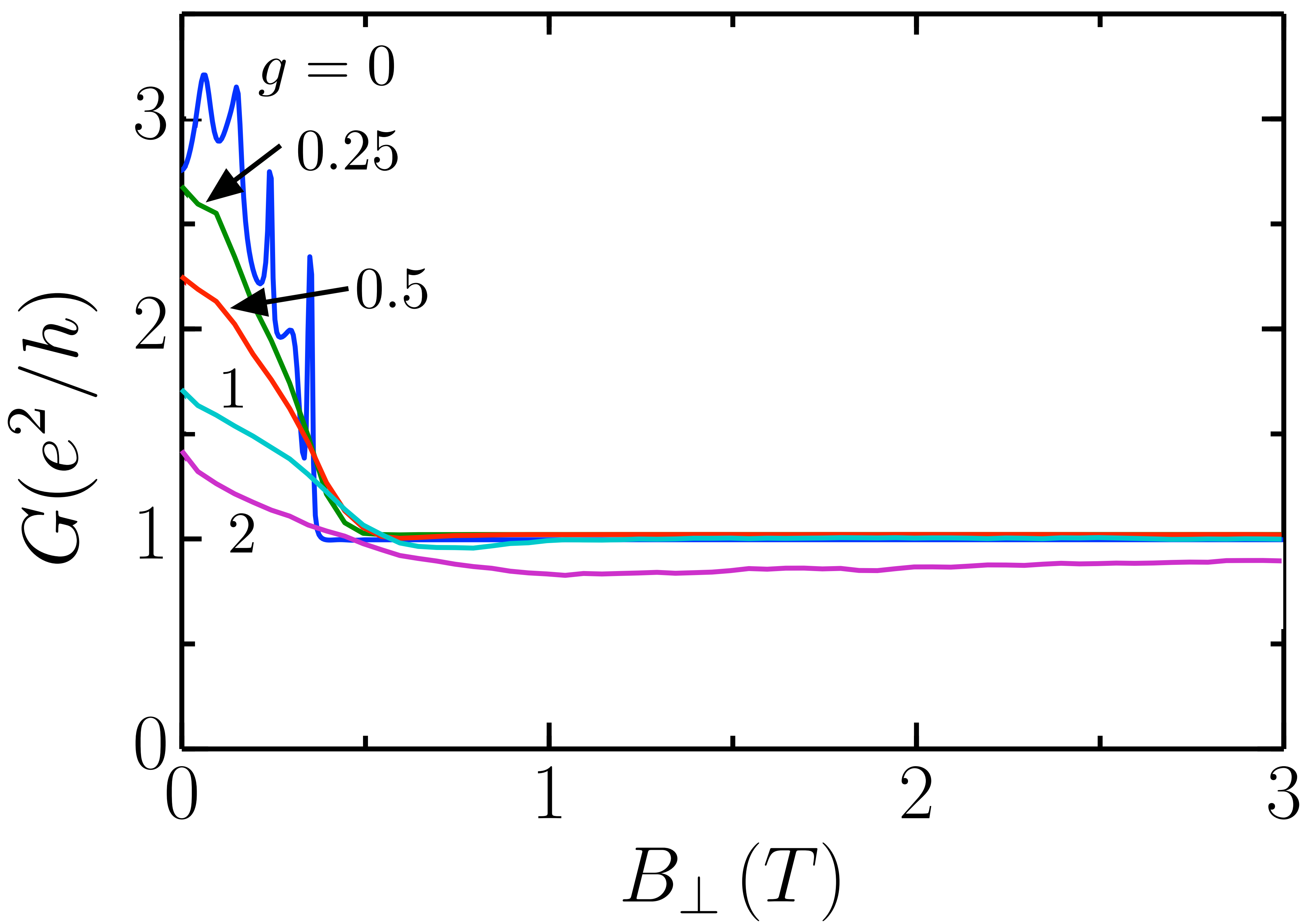
F. de Juan, R. Ilan, **JHB**, PRL 2014
Lee PRL 2009; Sitte et al PRL 2012

Conductance vs. chemical potential @ B = 2T



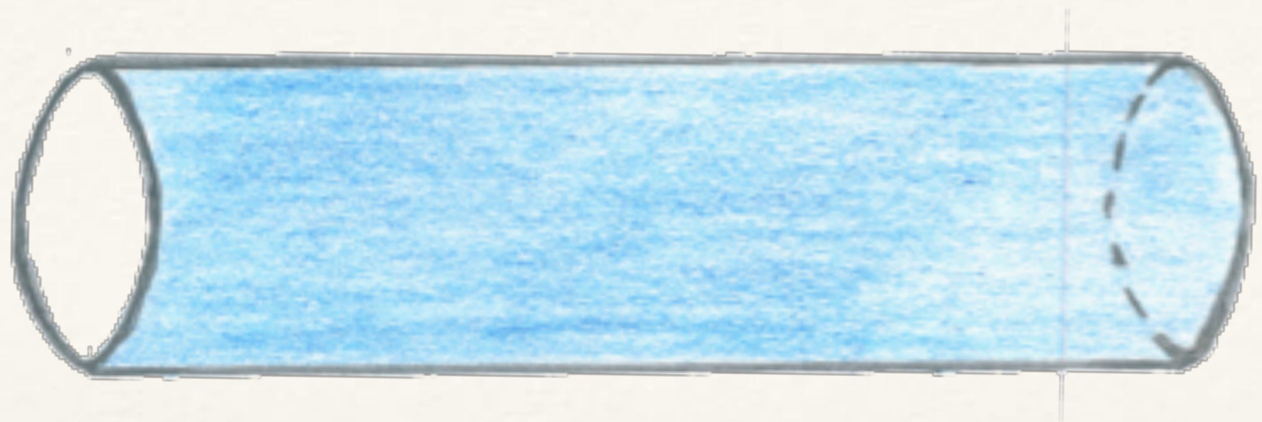
F. de Juan, R. Ilan, **JHB**, PRL 2014

Conductance vs. magnetic field at μ = 10 meV

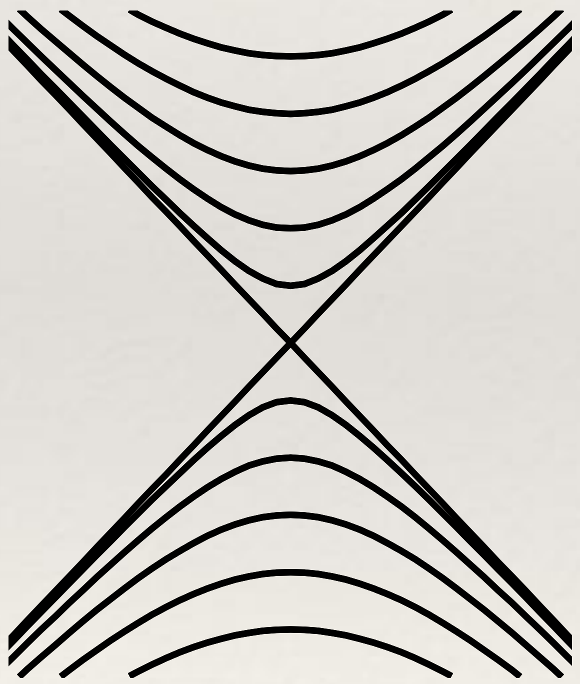


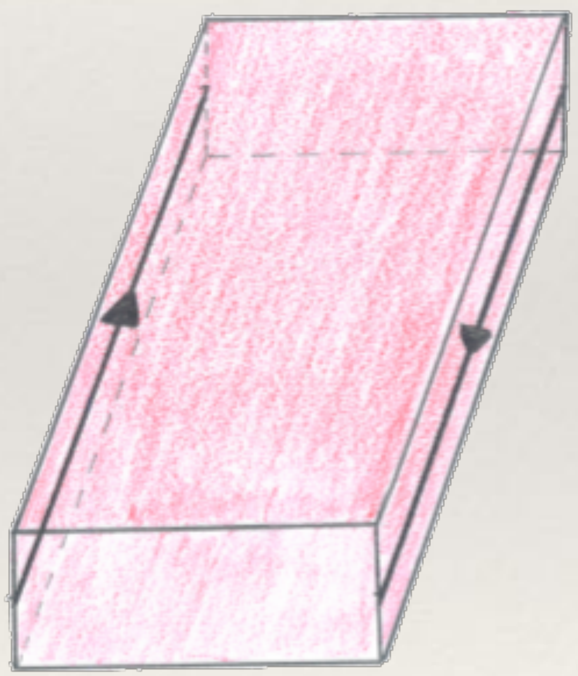
F. de Juan, R. Ilan, **JHB**, PRL 2014

Topological insulator nanowires host various interesting modes



Perfectly transmitted mode — Chiral modes — Majorana modes



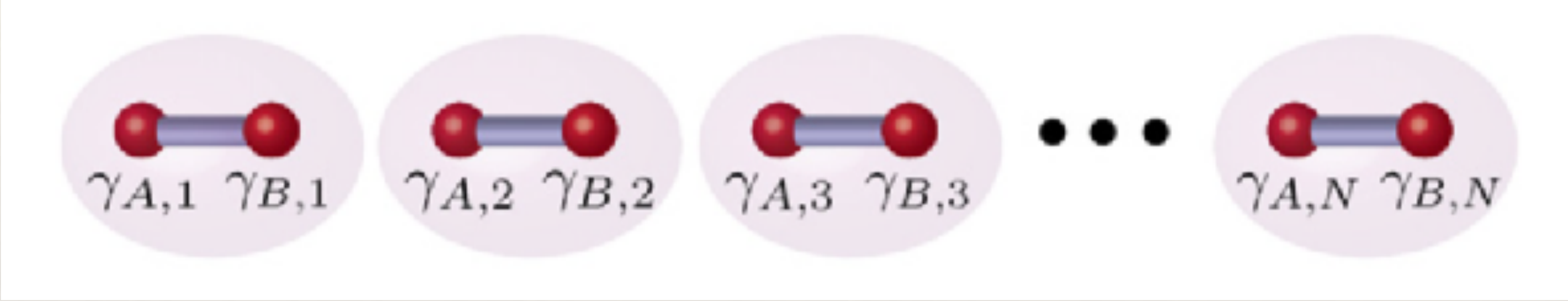


Kitaev chain

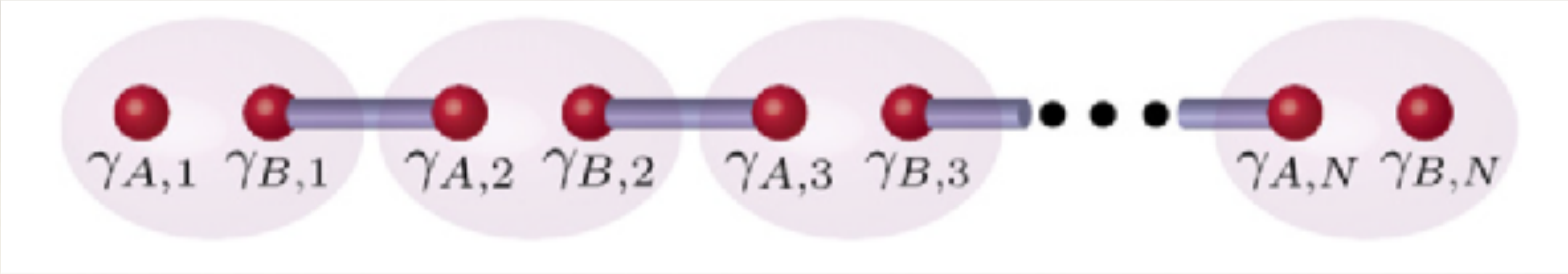
$$H = -\mu \sum_j c_j^\dagger c_j - \frac{1}{2} \sum_i (t c_j^\dagger c_{j+1} + \Delta e^{i\phi} c_j c_{j+1} + \text{H.c.})$$

$$c_j = \frac{e^{-i\phi/2}}{2}(\gamma_{B,j} + i\gamma_{A,j}) \qquad \gamma_{\alpha,j} = \gamma_{\alpha,j}^\dagger \qquad \{\gamma_{\alpha,j}, \gamma_{\alpha',j'}\} = 2\delta_{\alpha,\alpha'}\delta_{j,j'}$$

$\mu < 0; t = \Delta = 0$ $\qquad H = -\frac{\mu}{2} \sum_{j+1}^{N} (1 + i\gamma_{B,j}\gamma_{A,i})$



$\mu = 0; t = \Delta \neq 0$ $\qquad H = -i\frac{t}{2} \sum_{j=1}^{N-1} \gamma_{B,j}\gamma_{A,j+1}$



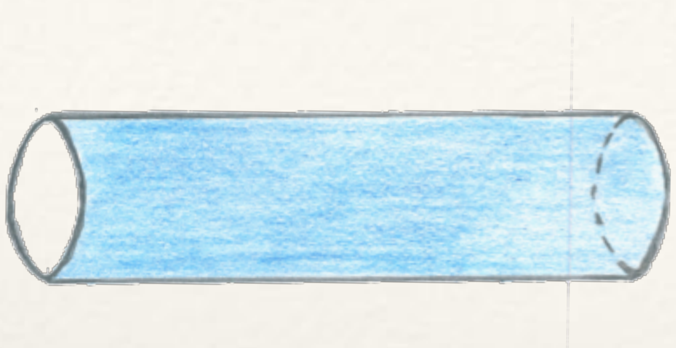
$$f = \frac{1}{2}(\gamma_{A,1} + i\gamma_{B,N})$$

$$f|0\rangle = 0 \qquad f^\dagger|0\rangle$$

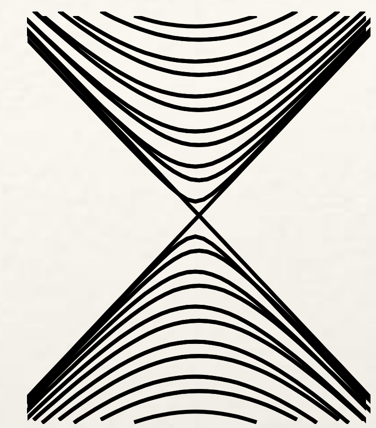
twofold degenerate ground state

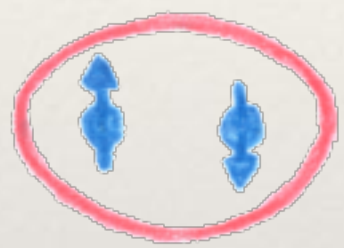
Alicea Rep. Prog. Phys. (2012)

A recipe for a tunable topological superconductor



+ $\phi = \frac{\phi_0}{2}$



+  (proximity)

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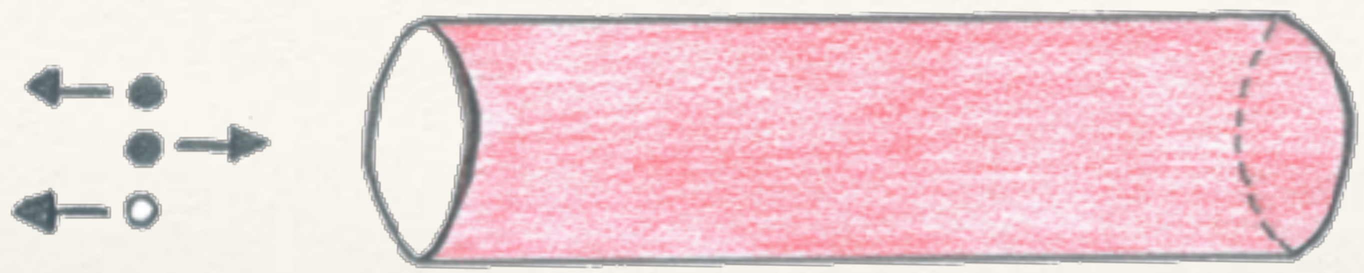
Cook and Franz PRB 2011

Detecting topological superconductivity via Béri degenercay

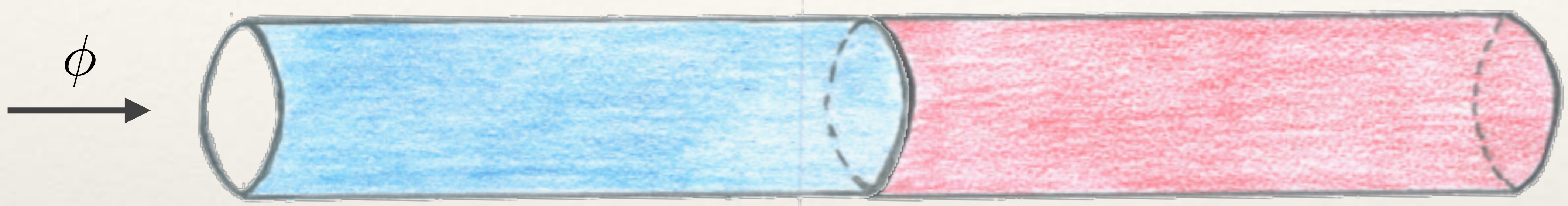


$$\sigma_x H^* \sigma_x = -H \qquad r^\dagger r = 1 \quad \underset{N=1}{\Rightarrow} \quad r_{ee} r_{he}^* = 0 \quad \Rightarrow \begin{cases} |r_{ee}| = 1 & G = 0 & \text{trivial} \\ |r_{he}| = 1 & G = \dfrac{2e^2}{h} & \text{topological} \end{cases}$$

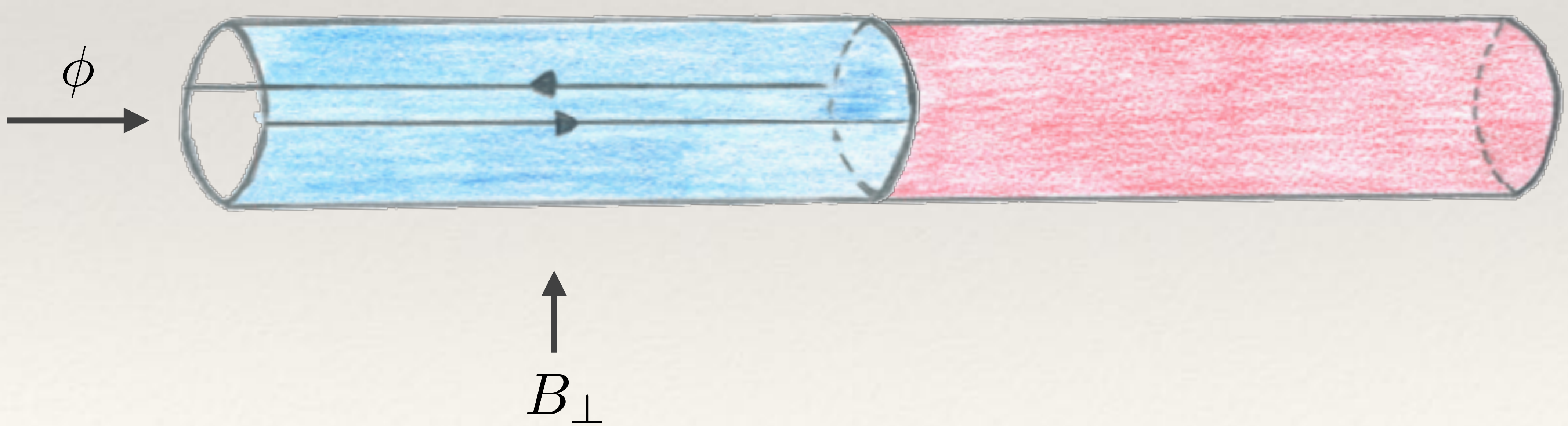
Béri PRB 2009; Wimmer, Akhmerov, Dahlhaus, Beenakker NJP (2011)

Getting to the single mode regime

Perfectly transmitted mode

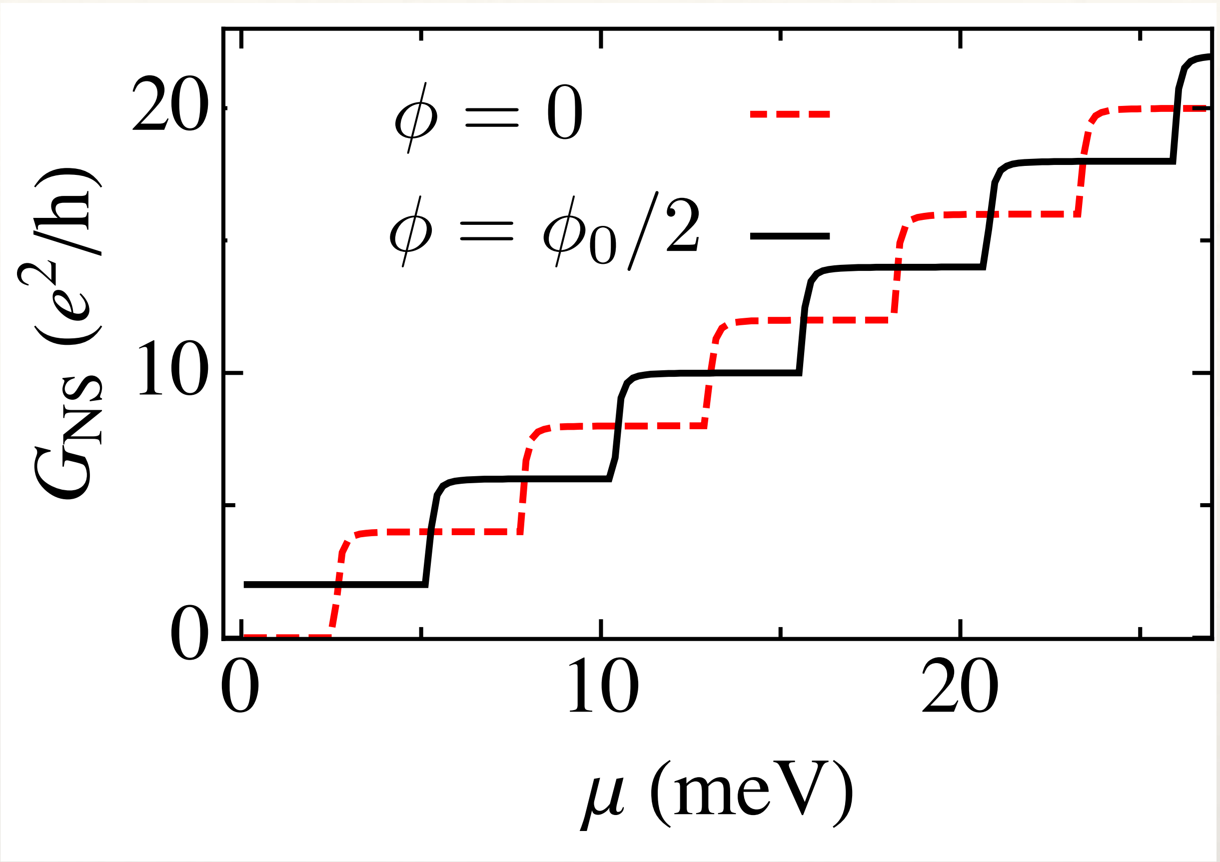


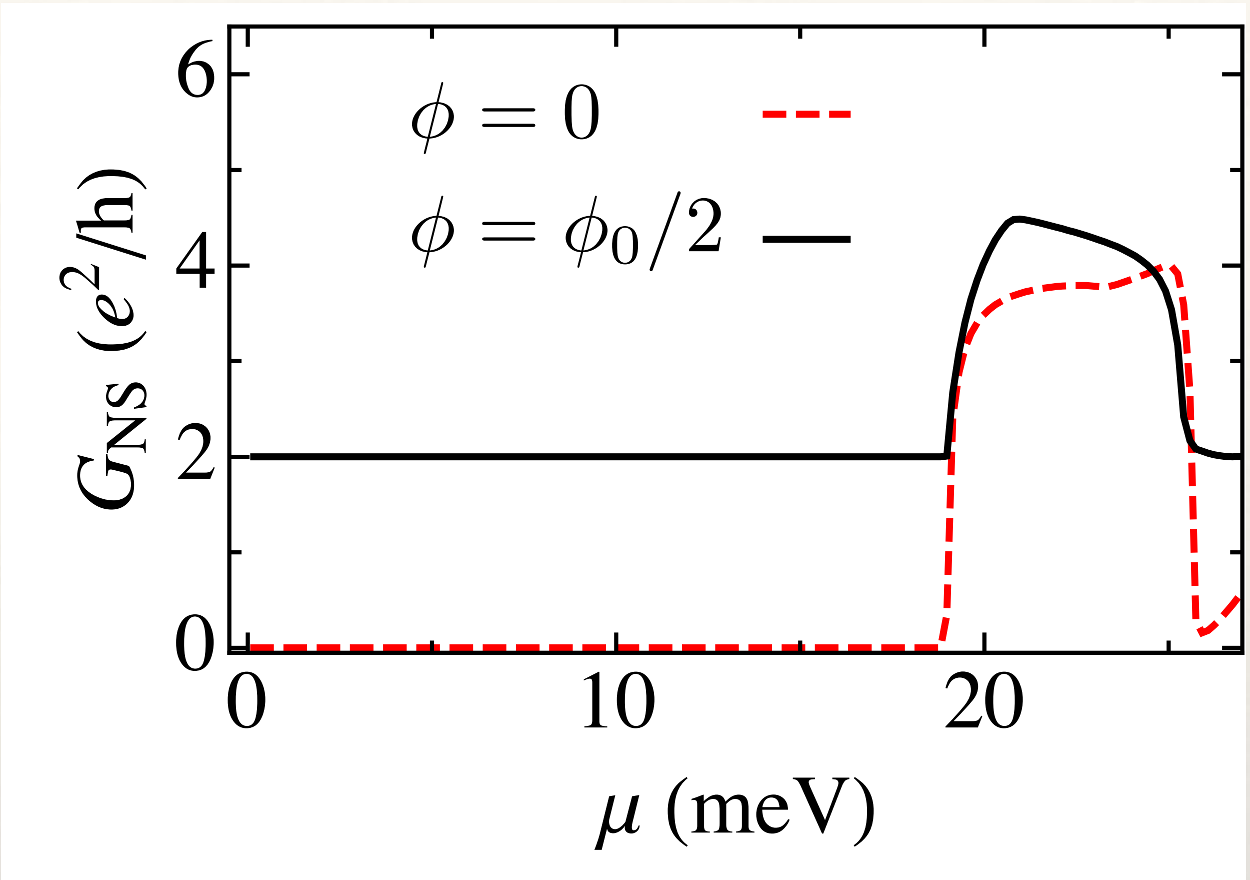
Chiral mode



F. de Juan, R. Ilan, **JHB**, PRL 2014

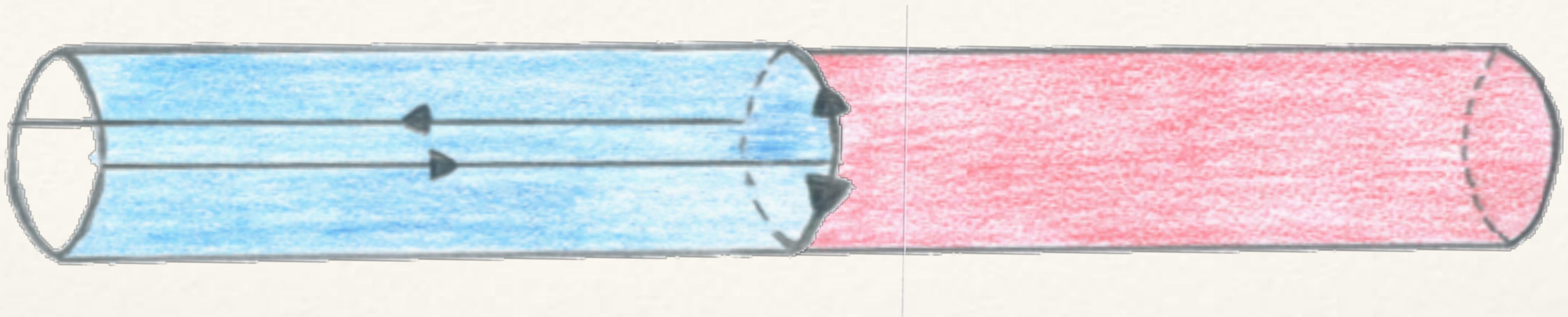
NS Conductance vs. chemical potential

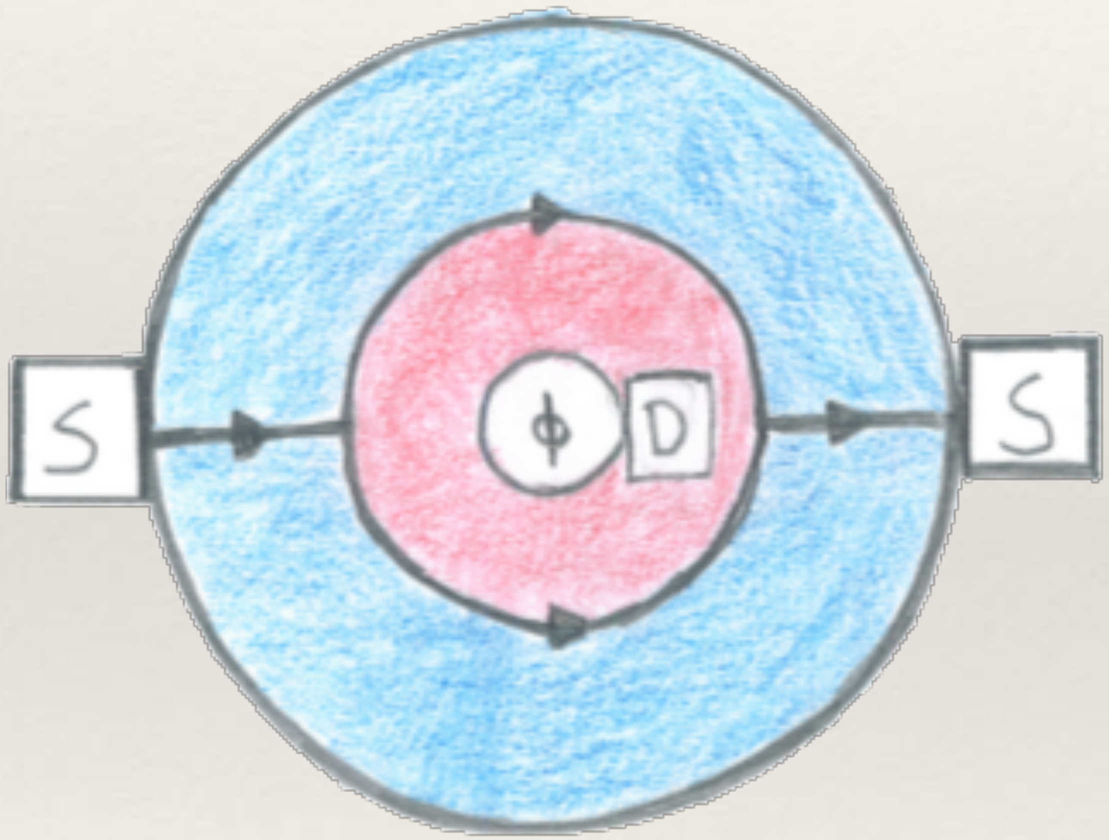


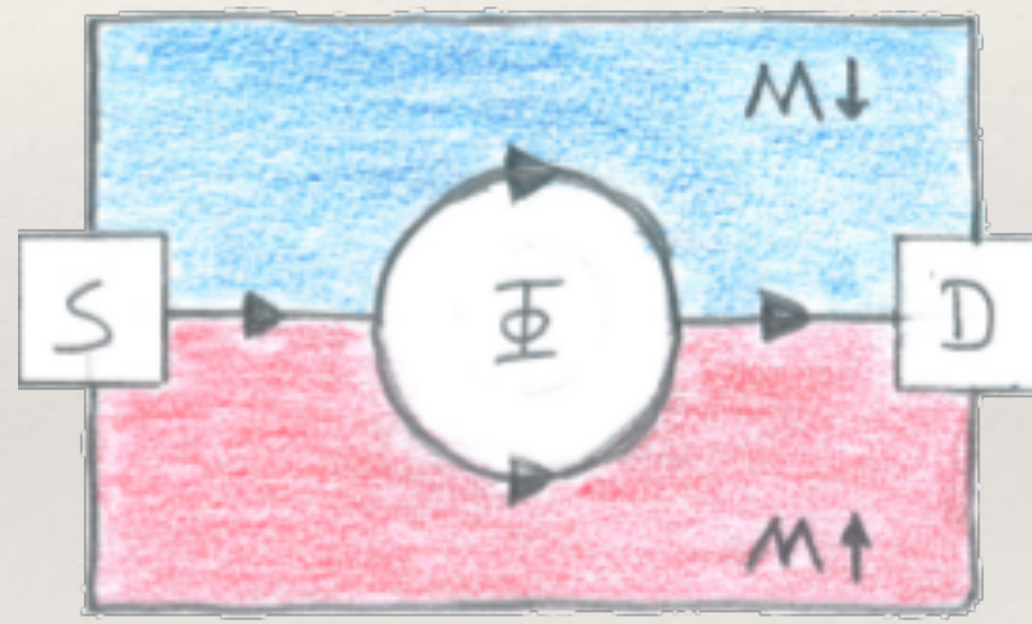


F. de Juan, R. Ilan, **JHB**, PRL 2014

Majorana interferometer



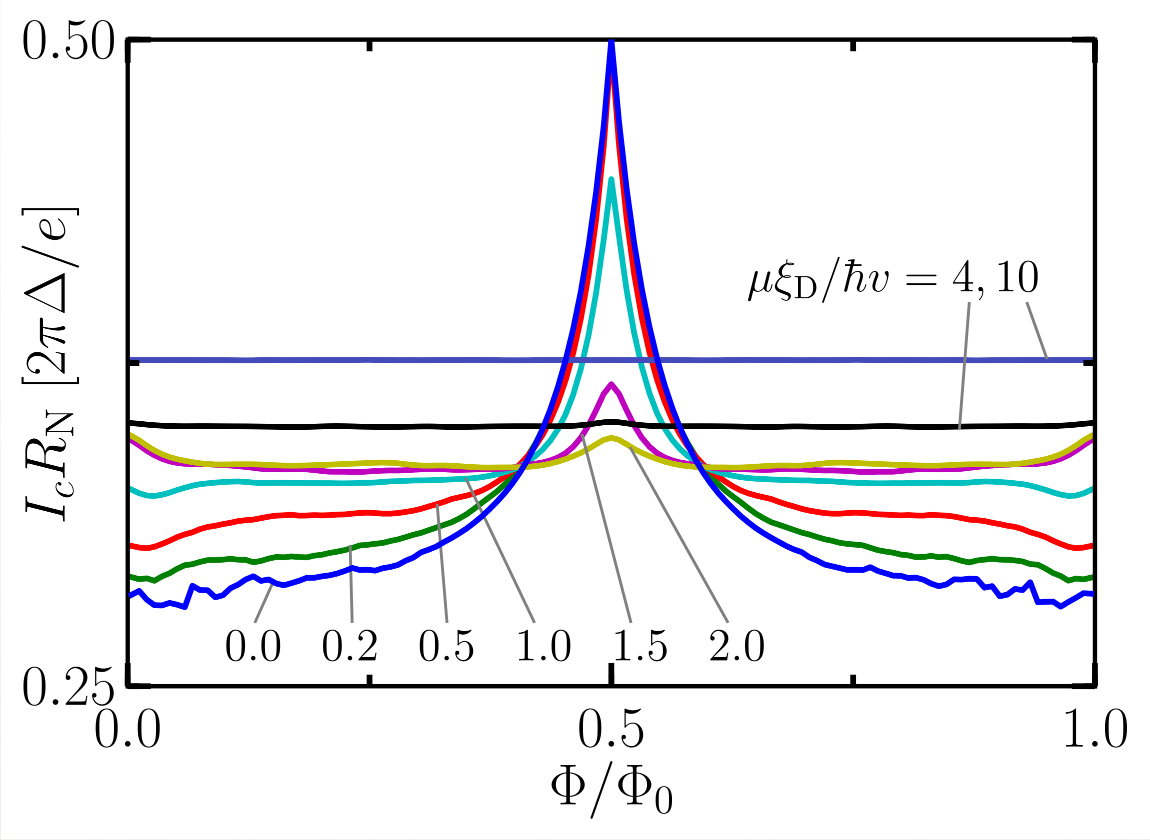


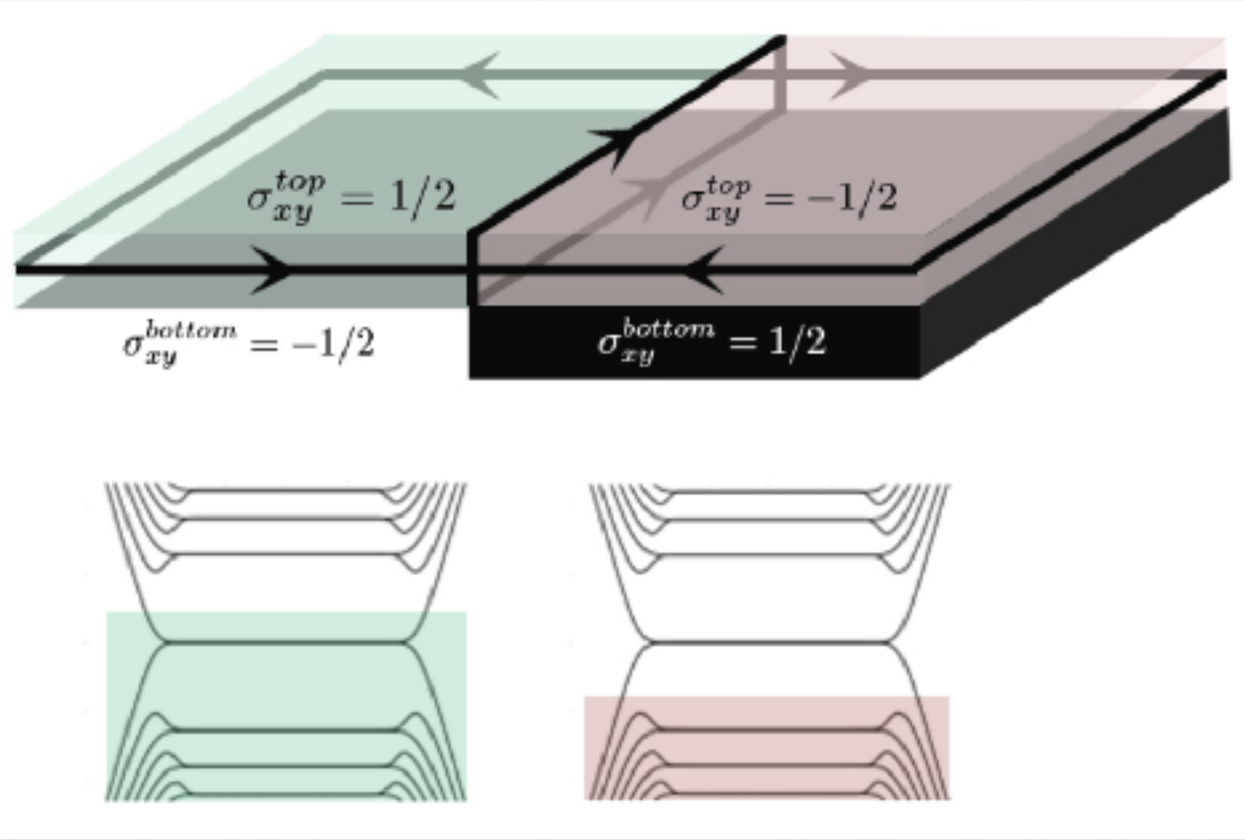


F. de Juan, R. Ilan, **JHB**, PRL 2014

Fu and Kane, PRL (2009), Akhmerov, Nilsson, and Beenakker PRL (2009)

SNS and p-n junctions

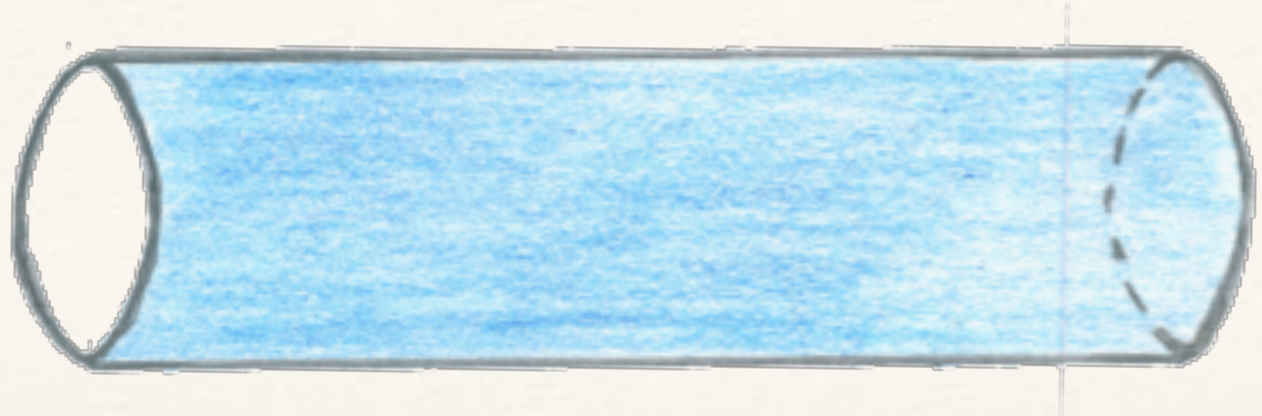




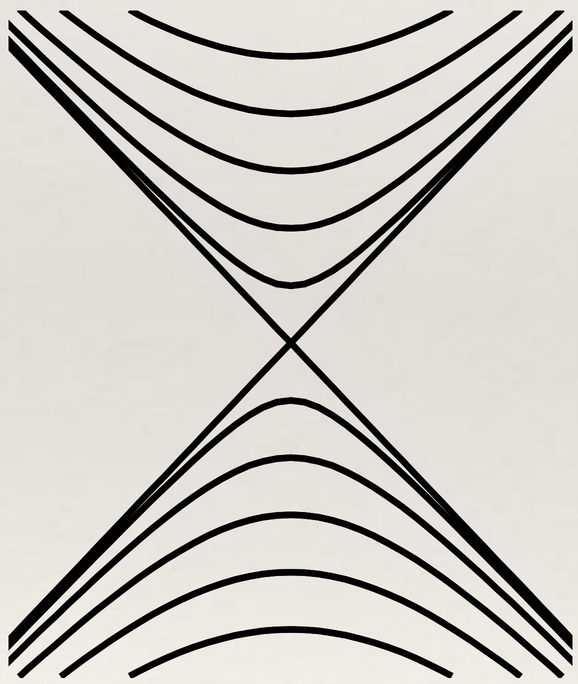
Ilan, JHB, Sim, Moore NJP 2014

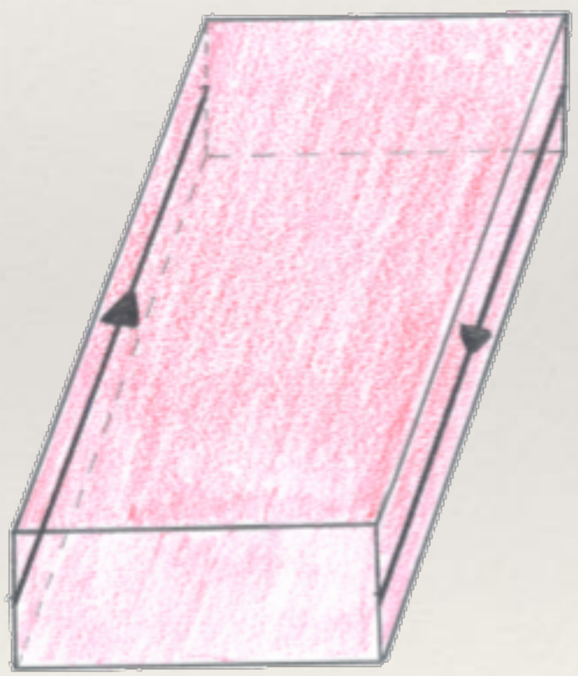
Ilan, de Juan, Moore PRL 2015

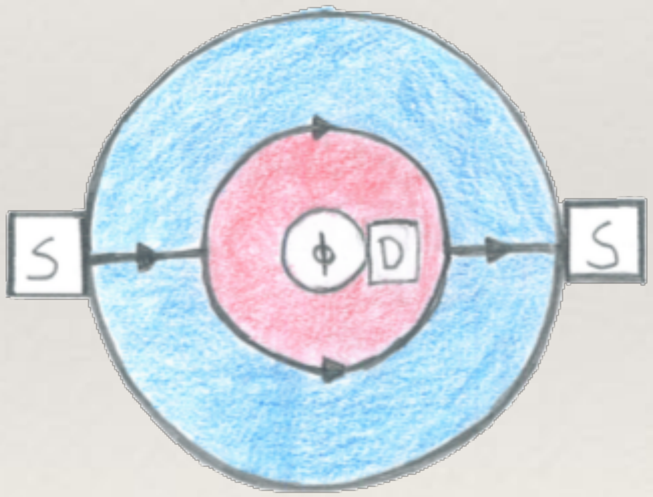
Topological insulator nanowires host various interesting modes



Perfectly transmitted mode — Chiral modes — Majorana modes







JHB, Brouwer, Moore PRL 2010; Ilan, **JHB**, Sim, Moore NJP 2014; F. de Juan, R. Ilan, **JHB**, PRL 2014

Review: **JHB** and Moore, Rep. Prog. Phys. 2013